Characterization of classes of functions by polynomial approximation

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1. Introduction

Let $f(x)$ be a 2π-periodic, continuous function with the Fourier series

$$S(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x),$$

and let $\tilde{f}(x)$ be the trigonometric conjugate of $f(x)$. Set

$$R_n(r; f) = \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1}\right)^r\right] A_k.$$

The following theorems hold:

(A) $f \in \mathrm{Lip}\, 1$ *if and only if* $\|\tilde{f} - R_n(1; \tilde{f})\|_C = O\left(\frac{1}{n}\right)$,

(B) $f' \in \mathrm{Lip}\, 1$ *if and only if* $\|f - R_n(2; f)\|_C = O\left(\frac{1}{n^2}\right)$.

(A) was proved long ago by ALEXITS [1] and ZAMANSKY [5], (B) by ZAMANSKY [5].

More generally, *if $f(x)$ has $r-1$ continuous derivatives* ($r \geqq 1$), *we have*
(C) $f^{(r-1)} \in \mathrm{Lip}\, 1$ *if and only if*

$$\|f - R_n(r; f)\|_C = O\left(\frac{1}{n^r}\right) \quad \textit{for } r \textit{ even}, \qquad \|\tilde{f} - R_n(r; \tilde{f})\|_C = O\left(\frac{1}{n^r}\right) \quad \textit{for } r \textit{ odd}.$$

This was also proved by ZAMANSKY [5], but the "if" part was given earlier by ZYGMUND [6].

The fact that for odd r we have to consider the approximation of the conjugate function $\tilde{f}(x)$ instead of $f(x)$, is inconvenient both from theoretical and practical point of view. Hence, it is natural to raise the following

Problem. Does there exist a sequence of linear operators $\{T_n\}$ such that $T_n f$ is a trigonometric polynomial or order n and, in the space $L^p(1 \leqq p \leqq \infty)$, the approximation property

$$\|f-T_n f\|_p = O\left(\frac{1}{n^r}\right) \tag{1}$$

is necessary and sufficient for $f^{(r-1)} \in \text{Lip}\,(1, p)$? [1])

The answer is yes. This was proved by TRIGUB [8] who introduced, to this aim, a special summation process of the Fourier series, somewhat similar to the Rogosinski means. Recently, his result was generalized by ŽUK [9].

Now, we shall indicate a general method to form polynomial operators for solving similar problems. Our way has the advantage that the results can be localized easily; this fact seems to have some interest, because it offers access to the characterization by algebraic polynomials of non-periodical functions belonging to the class Lip 1 in the interior of an arbitrary interval (a, b).

2. Trigonometric operators which characterize the classes Lip $(1, p)$

We call T_n a trigonometric operator of order n if $T_n f$ is a trigonometric polynomial of order n. By Z_p we denote the L^p-Zygmund class, i.e. the class of functions satisfying the condition

$$\|f(x+h)+f(x-h)-2f(x)\|_p = O(|h|).$$

We have the following general

Theorem 1. *Let $U_n f$ and $V_n f$ be trigonometric operators of order n such that*

$$\|f-U_n f\|_p = O\left(\frac{1}{n^r}\right) \tag{2}$$

for $f^{(r-1)} \in Z_p$ and that the relation

$$\|V_n^{(r)} f\|_p = O(1) \tag{3}$$

is equivalent to $f^{(r-1)} \in \text{Lip}\,(1, p)$, where $V_n^{(r)} f$ denotes the r th derivative of $V_n f$. Setting

$$T_n f = U_n f + \frac{1}{n^r} V_n^{(r)} f, \tag{4}$$

condition (1) *is both necessary and sufficient for $f^{(r-1)} \in \text{Lip}\,(1, p)$.*

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[1]) Of course, we assume that Lip $(1, \infty)$ is the ordinary Lip 1 class and the ∞-norm is the C norm for continuous $f(x)$. Also, $f \in \text{Lip}\,(1, 1)$ is equivalent to $f \in BV$.

Proof. Suppose $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$. Then (2) and (3) follow by assumption, and hence (1) is an immediate consequence of definition (4).

Inversely, suppose (1) satisfied. Then by the well-known Bernstein—Zygmund theorem (generalized to L^p) it follows that $f^{(r-1)} \in Z_p$; hence (2) holds good. So we get by (1), (2) and (4)

$$\frac{1}{n^r} \|V_n^{(r)} f\|_p = O\left(\frac{1}{n^r}\right);$$

therefore (3) is valid, too. But this is equivalent by assumption to $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$, and so our theorem is proved.

Corollary. *Setting*

$$T_n f = 2\sigma_{2n-1}(f) - \sigma_{n-1}(f) + \frac{1}{n^r}\, \sigma_n^{(r)}(f),$$

where $\sigma_m(f)$ denotes the mth Fejér mean of the Fourier series $S(f)$, we get a trigonometric operator of order $2n-1$ for which (1) *is equivalent to* $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$.

Indeed, $U_n f = 2\sigma_{2n-1}(f) - \sigma_{n-1}(f)$ is the nth de la Vallée Poussin mean which satisfies (2), furthermore, as well known (cf. ZYGMUND [7], chapter IV), condition (3) is equivalent for $V_n f = \sigma_n(f)$ to $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$.

By this corollary our problem could be considered as essentially solved, the only lack is that the operator T_n defined in the corollary is not a trigonometric polinomial of order n, but of order $2n-1$. We can easily help there.

Theorem 2. *By choosing*

$$T_n f = R_n\left(r + \frac{1-(-1)^r}{2}; f\right) + \frac{1-(-1)^r}{2n^r} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) k^r B_k, \tag{5}$$

where $B_k(x) = a_k \sin kx - b_k \cos kx$, condition (1) *is both necessary and sufficient for $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$, where $1 \leqq p \leqq \infty$.*

If r is even, then $T_n f = R_n(r; f)$ and our statement is already proved by theorem (C). Hence we have to consider only the case of r odd. But then

$$T_n f = R_n(r+1; f) + \frac{1}{n^r} \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) k^r B_k. \tag{6}$$

Setting $U_n f = R_n(r+1; f)$, it is easily seen that $U_n f$ satisfies condition (2) if $f^{(r-1)} \in Z_p$ (cf. ZYGMUND [6] and ALJANČIĆ [2]). Now

$$V_n^{(r)} f = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) k^r B_k$$

is, for odd r, the r times differentiated nth Fejér mean, therefore (cf. ZYGMUND [7], chapter IV), condition (3) is equivalent to $f^{(r-1)} \in \mathrm{Lip}\,(1, p)$. All the conditions of Theorem 1 being satisfied, our statement is a consequence of this theorem.

3. Localization and approximation by algebraic polynomials

We say that the function $f(x)$ satisfies the Lipschitz condition in the open interval (a, b), if $f \in \mathrm{Lip}\, 1$ in every closed interval $[a+\varepsilon, b-\varepsilon] \subset (a, b)$, the Lipschitz constant being dependent of the choice of $\varepsilon > 0$. We suppose $(a, b) \subset (-\pi, \pi)$ and set $f(x)=0$ for $x \in (-\pi, \pi)-(a, b)$. By $S(f)$ we understand the Fourier series of the function $f(x)$ extended in such a way to $(-\pi, \pi)$ and then periodically to the whole straight line.

For the study of local approximation we introduce the Riesz means of type r and order r of the series $S(f)$:

$$R_n^*(r; f) = \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1}\right)^r\right]^r A_k.$$

Put

$$T_n f = R_n^*\left(r + \frac{1-(-1)^r}{2}; f\right) + \frac{1-(-1)^r}{2n^r} \sum_{k=0}^{n} \frac{A_{n-k}^r}{A_n^r} k^r B_k, \tag{5a}$$

wehre $A_m^l = \binom{m+l}{m}$.

Theorem 3. *By choosing T_n as in* (5a), *we have $f^{(r-1)} \in \mathrm{Lip}\, 1$ in (a, b) if and only if*

$$\|f - T_n f\|_{C(a+\varepsilon, b-\varepsilon)} = O\left(\frac{1}{n^r}\right) \tag{7}$$

for every $\varepsilon > 0$, where the constant in the O-sign may depend on ε.

Proof. Suppose first $f^{(r-1)} \in \mathrm{Lip}\, 1$ in (a, b). Then, for r even, $T_n f$ reduces to $R_n^*(r; f)$ and (7) follows by the train of thoughts of SUNOUCHI [4b]. If r is odd, $T_n f$ has a similar form as in (6). We get then

$$\|f - R_n^*(r+1; f)\|_{C(a+\varepsilon, b-\varepsilon)} = O\left(\frac{1}{n^r}\right); \tag{8}$$

further we also have (SUNOUCHI [4a])

$$\left\| \sum_{k=1}^{n} \frac{A_{n-k}^r}{A_n^r} k^r B_k \right\|_{C(a+\varepsilon, b-\varepsilon)} = O(1), \tag{9}$$

and then (7) follows from (6), (8) and (9).

Suppose now (7) is satisfied. $T_n f$ being a trigonometric polynomial of order n, it follows by the localized Bernstein—Zygmund theorem that $f^{(r-1)}(x)$ exists and belongs to the Zygmund class in $[a+2\varepsilon, b-2\varepsilon]$. Therefore

$$\|f - R_n^*(r+1; f)\|_{C(a+3\varepsilon, b-3\varepsilon)} = O\left(\frac{1}{n^r}\right),$$

and so we have also

$$\frac{1}{n^r}\left\|\sum_{k=1}^{n}\frac{A_{n-k}^r}{A_n^r}k^r B_k\right\|_{C(a+3\varepsilon,\, b-3\varepsilon)} = O\left(\frac{1}{n^r}\right).$$

From the last evaluation it follows by the above mentioned result of SUNOUCHI [4a] that $f^{(r-1)}\in \text{Lip } 1$ in $[a+3\varepsilon, b-3\varepsilon]$. Since $\varepsilon>0$ is arbitrary, this means $f^{(r-1)}\in \text{Lip } 1$ in the open interval (a, b), as we have stated.

We are now able also to characterize non-periodical functions satisfying the Lipschitz condition by their approximation with algebraic polynomials. Denote to this aim by $t_n(x)$ and $u_n(x)$ the nth normed Chebyshev polynomials of the first and second kind. Set

$$a_n = \int_{-1}^{1} f(x)t_n(x)\frac{dx}{\sqrt{1-x^2}},$$

where $f(x)=0$ outside some interval (a, b) with $-1<a<b<1$.

Theorem 4. *Set*

(10) $$T_n f = \sum_{k=0}^{n}\left[1-\left(\frac{k}{n+1}\right)^2\right]^2 a_k t_k + \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)k a_k u_k.$$

The function $f(x)$ belongs to Lip 1 *in (a, b) if and only if we have for every $\varepsilon>0$*

(11) $$\|f-T_n f\|_{C(a+\varepsilon,\, b-\varepsilon)} = O\left(\frac{1}{n}\right).$$

Proof. Introducing the variable $\theta=\arccos x$, the function $f(x)$ is transformed in $f^*(\theta)=f(\cos\theta)$, the operator $T_n f$ in

$$T_n f^* = \sum_{k=0}^{n}\left[1-\left(\frac{k}{n+1}\right)^2\right]^2 a_k \cos k\theta + \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n+1}\right)k a_k\frac{\sin k\theta}{\sin\theta},$$

and the open interval (a, b) in $(a', b')\subset(0, \pi)$. The first sum on the right hand side of (11) is $R_n^*(2; f^*)$, while the second equals $\frac{1}{n\sin\theta}\sigma_n'(f^*)$. For $\theta\in[a'+\delta, b'-\delta]$ with $\delta>0$ we have $0<\alpha\leqq\sin\theta\leqq 1$, therefore the last expression and $n^{-1}\cdot\sigma_n'(f^*, \theta)$ are $=O(n^{-1})$ at the same time, i.e. (11) is equivalent to

(12) $$\|f^*-T_n^* f^*\|_{C(a'+\delta,\, b'-\delta)} = O\left(\frac{1}{n}\right),$$

where $\delta>0$ is arbitrary and $T_n^* f^*$ is defined by

$$T_n^* f^* = R_n^*(2; f^*) + \frac{1}{n}\sigma_n'(f^*).$$

But, by Theorem 3, (12) is equivalent to the statement that $f^*(\theta)=f(\cos\theta)$ satisfies the Lipschitz condition in the open interval (a', b'). Returning to the variable $x=\cos\theta$ and taking into account that $f^*(\theta)$ in (a', b') and $f(x)$ in (a, b) satisfy the Lipschitz condition at the same time, our theorem is entirely proved.

Remarks. 1. It could be proved that, using the operator $T_n f$ defined in (5a), the condition

$$\|f-T_n f\|_{p(a,b)} = O\left(\frac{1}{n^r}\right)$$

is necessary and sufficient for $f^{(r-1)}\in \mathrm{Lip}\,(1, p)$ in the open interval $(a, b)\subset(-\pi, \pi)$. Hereby the L^p norm has to be taken over every interval $[a+\varepsilon, b-\varepsilon]\subset(a, b)$.

2. Let $\{p_n(x)\}$ be the system of orthonormal polynomials belonging to a weight function $w(x)\geqq 0$ having the property $0<m\leqq w(x)\leqq M$ for $x\in[a, b]\subset[-1, 1]$. Denoting by $\sigma_n(x)$ the nth Fejér mean of the expansion

$$f(x)\sim\sum_{k=0}^{\infty} c_k p_k(x),$$

one can prove by a theorem of G. Freud [3] that the Lebesgue constants of the de la Vallée Poussin delayed means $2\sigma_{2n-1}-\sigma_{n-1}$ are bounded in every subinterval $[a+\varepsilon, b-\varepsilon]$ of (a, b). Then, setting $T_n f = 2\sigma_{2n-1}-\sigma_{n-1}+n^{-1}\cdot\sigma_n'$, one can prove easily that $f\in\mathrm{Lip}\,(1, p)$ in the open interval (a, b) if and only if

$$\|f-T_n f\|_{p(a,b)} = O\left(\frac{1}{n}\right) \quad (1\leqq p\leqq\infty),$$

where the L^p norm has to be taken over every closed subinterval of (a, b).

3. We were concerned with the case $\mathrm{Lip}\,(1, p)$, because only the case $\alpha=1$ of Lipschitz classes $\mathrm{Lip}\,(\alpha, p)$ is critical. But it can be seen easily that, in the theorems 2 and 3, we may substitute $\mathrm{Lip}\,(\alpha, p)$ for $\mathrm{Lip}\,(1, p)$, if we substitute in the statements $O(n^{-r+1-\alpha})$ for $O(n^{-r})$. Thus defined as in (5), then $f^{(r-1)}\in\mathrm{Lip}\,(\alpha, p)$ is equivalent to $\|f-T_n f\|_p = O(n^{-r+1-\alpha})$ for every positive $\alpha\leqq 1$ and $r=1, 2, \ldots$.

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