On $|\overline{N}, p_n|$ summability factors

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1. Introduction

1. Introduction. Let Σa_n be an infinite series with the partial sums S_n. Let $\{p_n\}$ be a sequence of constants, real or complex, with partial sums $\{P_n\}$ and $P_{-1}=p_{-1}=0$. The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu \quad (P_n \neq 0) \tag{1. 1}$$

defines the $(\overline{N}, p_n)$ means of the sequence $\{S_n\}$, or of the series Σa_n, generated by the sequence of constants $\{p_n\}$. If $\lim_{n\to\infty} t_n$ exists, we say that the series Σa_n is summable $(\overline{N}, p_n)$ [1]; and if the sequence $\{t_n\}$ is of bounded variation, that is

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty, \tag{1. 2}$$

then the series Σa_n is said to be absolutely summable $(\overline{N}, p_n)$, or simply summable $|\overline{N}, p_n|$.

The conditions of regularity of the method of summability $(\overline{N}, p_n)$ defined by (1. 1) are

$$\lim_{n\to\infty} |P_n| = \infty \tag{1. 3}$$

and

$$\sum_{\nu=0}^{n} |p_\nu| = O(|P_n|). \tag{1. 4}$$

If $\{p_n\}$ is real and non-negative, (1. 4) is automatically satisfied, and then (1. 3) is a necessary and sufficient condition for the regularity of the method. It is known that $|\overline{N}, p_n|$ is equivalent to $|R, P_n, 1|^*$, where $|R, P_n, 1|^*$ is a discrete Riesz mean of order one and type P_n.

In the particular case when $p_n=1$, the $(\overline{N}, p_n)$ mean reduces to the familiar $(C, 1)$ mean. Also when $p_n=e^n$, $|\overline{N}, p_n|$ is equivalent to $|C, 0|$.

2. The sequence $\{\varepsilon_n\}$ is said to be a summability factor of the series Σa_n for a summability method Q, if $\Sigma a_n \varepsilon_n$ is summable by the method Q whereas, in general, Σa_n need not be summable. The summability factors for absolute Cesàro methods of summation were obtained by KOGBETLIANTZ [2]. He proved the following theorem.

Theorem. *If Σa_n is $|C, \delta|$, then $\sum \frac{a_n}{n^{\delta-\gamma}}$ is summable $|C, \gamma|$ for $\gamma \leqq \delta$, $\gamma, \delta > 0$.*

MOHANTY [3] has shown that whenever Σa_n is $|R, \log n, 1|$, the series $\sum \frac{a_n}{\log n}$ is $|C, 1|$ (see also TATCHELL [4]).

The object of this paper is to establish the following theorem, which is an analogue of the theorem of KOGBETLIANTZ and includes, amongst others, the above result of MOHANTY.

Theorem. *If Σa_n is $|\bar{N}, p_n|$ summable, then $\sum \frac{a_n p_n Q_n}{q_n P_n}$ is $|\bar{N}, q_n|$ summable provided $\{p_n\}$ and $\{q_n\}$ are positive sequences such that the sequences $\left\{\Delta\left(\frac{Q_n}{q_n}\right)\right\}$, $\left\{\frac{Q_n p_n}{P_n q_n}\right\}$ and $\left\{\frac{Q_{n+1}}{q_{n+1}} \cdot \frac{\Delta p_n}{p_n}\right\}$ are bounded.*

Proof. Let t_n denote the $(\bar{N}, p_n)$ mean of Σa_n. Then

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} (P_n - P_{\nu-1}) a_\nu .$$

If τ_n denotes the $(\bar{N}, q_n)$ mean of $\sum \frac{a_n p_n Q_n}{q_n P_n}$, we similarly have

$$\tau_n = \frac{1}{Q_n} \sum_{\nu=0}^{n} (Q_n - Q_{\nu-1}) \frac{a_\nu p_\nu Q_\nu}{q_\nu P_\nu} = \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_\nu}{P_\nu q_\nu} - \frac{1}{Q_n} \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu}$$

and

$$\tau_{n+1} = \sum_{\nu=0}^{n+1} \frac{a_\nu p_\nu Q_\nu}{P_\nu q_\nu} - \frac{1}{Q_{n+1}} \sum_{\nu=0}^{n+1} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu}.$$

Hence

$$\tau_{n+1} - \tau_n = \frac{a_{n+1} p_{n+1} Q_{n+1}}{P_{n+1} q_{n+1}} + \left(\frac{1}{Q_n} - \frac{1}{Q_{n+1}}\right) \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} - \frac{a_{n+1} p_{n+1} Q_n Q_{n+1}}{P_{n+1} q_{n+1} Q_{n+1}} =$$

$$= \frac{a_{n+1} p_{n+1}}{P_{n+1}} + \left(\frac{q_{n+1}}{Q_n Q_{n+1}}\right) \sum_{\nu=0}^{n} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} = \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n+1} \frac{a_\nu p_\nu Q_{\nu-1} Q_\nu}{P_\nu q_\nu} =$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \Delta \left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) s_\nu + \frac{q_{n+1}}{Q_n Q_{n+1}} \cdot \frac{Q_n Q_{n+1} p_{n+1} s_{n+1}}{P_{n+1} q_{n+1}} =$$

$$= - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{1}{p_\nu} \Delta \left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta P_{\nu-1} t_{\nu-1} - \frac{\Delta P_n t_n}{P_{n+1}} =$$

$$= -\frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{1}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \{t_{\nu-1} \Delta P_{\nu-1} + P_\nu \Delta t_{\nu-1}\} - \frac{t_n \Delta P_n + P_{n+1} \Delta t_n}{P_{n+1}} =$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) t_{\nu-1} - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} +$$

$$+ \frac{p_{n+1} t_n}{P_{n+1}} - \Delta t_n.$$

Further,

$$\sum_{\nu=0}^{n} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) t_{\nu-1} = -\sum_{\nu=0}^{n-1} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} - \frac{Q_n Q_{n+1} p_{n+1} t_{n-1}}{P_{n+1} q_{n+1}}.$$

Consequently,

$$\tau_{n+1} - \tau_n = -\frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n-1} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} - \frac{p_{n+1} \Delta t_{n-1}}{P_{n+1}} -$$

$$- \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} - \Delta t_n.$$

Or,

$$\tau_n - \tau_{n+1} = \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} +$$

$$+ \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} + \Delta t_n.$$

Hence,

$$\sum_{n=0}^{\infty} |\tau_n - \tau_{n+1}| \leqq \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} \right| +$$

$$+ \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} \right| + \sum_{n=0}^{\infty} |\Delta t_n| = \Sigma_1 + \Sigma_2 + \Sigma_3 \quad \text{(say)}.$$

Now,

$$\Sigma_1 = \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} \Delta t_{\nu-1} \right| \leqq$$

$$\leqq \sum_{\nu=0}^{\infty} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| \sum_{n=\nu}^{\infty} \left(\frac{1}{Q_n} - \frac{1}{Q_{n+1}}\right) = \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| < \infty,$$

by hypothesis, and also $\Sigma_3 < \infty$ by hypothesis. Now,

$$\left| \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1} Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1} \right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1} Q_\nu}{q_\nu}\right) + \frac{Q_{\nu+1} Q_\nu}{q_{\nu+1}} \left(1 - \frac{P_\nu p_{\nu+1}}{P_{\nu+1} p_\nu}\right) \right\} \Delta t_{\nu-1} \right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1}Q_\nu}{q_\nu}\right) + \frac{Q_{\nu+1}Q_\nu}{q_{\nu+1}} \left(\frac{p_\nu P_{\nu+1} - P_\nu p_{\nu+1}}{P_{\nu+1}p_\nu}\right)\right\} \Delta t_{\nu-1}\right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1}Q_\nu}{q_\nu}\right) + \frac{Q_\nu Q_{\nu+1}}{q_{\nu+1}} \left(\frac{p_\nu (P_{\nu+1} - P_\nu) + P_\nu \Delta p_\nu}{P_{\nu+1}p_\nu}\right)\right\} \Delta t_{\nu-1}\right| =$$

$$= \left| \sum_{\nu=0}^{n} \left\{ \Delta\left(\frac{Q_{\nu-1}Q_\nu}{q_\nu}\right) + \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1}q_{\nu+1}} + \frac{Q_\nu Q_{\nu+1}}{q_{\nu+1}} \cdot \frac{P_\nu}{P_{\nu+1}} \cdot \frac{\Delta p_\nu}{p_\nu}\right\} \Delta t_{\nu-1}\right| =$$

$$= \left| \sum_{\nu=0}^{n} - Q_\nu \Delta t_{\nu-1} + \sum_{\nu=0}^{n} Q_\nu \Delta\left(\frac{Q_\nu}{q_\nu}\right) \Delta t_{\nu-1} + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} P_\nu \Delta p_\nu}{q_{\nu+1} P_{\nu+1} p_\nu} \Delta t_{\nu-1} + \right.$$

$$\left. + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{q_{\nu+1} P_{\nu+1}} \Delta t_{\nu-1} \right| \leqq \sum_{\nu=0}^{n} Q_\nu |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} Q_\nu \left|\Delta\left(\frac{Q_\nu}{q_\nu}\right)\right| |\Delta t_{\nu-1}| +$$

$$+ \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{q_{\nu+1} P_{\nu+1}} |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} \left|\frac{\Delta p_\nu}{p_\nu}\right| |\Delta t_{\nu-1}| \frac{P_\nu Q_\nu Q_{\nu+1}}{P_{\nu+1} q_{\nu+1}}.$$

Hence,

$$\Sigma_2 = \sum_{n=0}^{n} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{\nu=0}^{n} \frac{P_\nu}{p_\nu} \Delta\left(\frac{Q_{\nu-1}Q_\nu p_\nu}{P_\nu q_\nu}\right) \Delta t_{\nu-1}\right| \leqq$$

$$\leqq \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left\{ \sum_{\nu=0}^{n} Q_\nu |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} Q_\nu \left|\Delta\left(\frac{Q_\nu}{q_\nu}\right)\right| |\Delta t_{\nu-1}| + \right.$$

$$\left. + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| + \sum_{\nu=0}^{n} \frac{Q_\nu Q_{\nu+1} P_\nu}{P_{\nu+1} q_{\nu+1}} \left|\frac{\Delta p_\nu}{p_\nu}\right| |\Delta t_{\nu-1}| \right\} =$$

$$= \sum_{\nu=0}^{\infty} |\Delta t_{\nu-1}| + \sum_{\nu=0}^{\infty} \left|\Delta\left(\frac{Q_\nu}{q_\nu}\right)\right| |\Delta t_{\nu-1}| + \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} p_{\nu+1}}{P_{\nu+1} q_{\nu+1}} |\Delta t_{\nu-1}| +$$

$$+ \sum_{\nu=0}^{\infty} \frac{Q_{\nu+1} P_\nu}{q_{\nu+1} P_{\nu+1}} \frac{|\Delta p_\nu|}{p_\nu} |\Delta t_{\nu-1}| = O(\Sigma |\Delta t_{\nu-1}|) = O(1).$$

Hence, $\Sigma |\tau_n - \tau_{n+1}| < \infty$. This establishes the theorem.

3. It is easy to see that, by taking $p_n = \frac{1}{n+1}$ and $q_n = 1$, the result of MOHANTY follows from our theorem.

References

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