On Haar and Schauder series

By L. G. PÁL and F. SCHIPP in Budapest

1. Consider the Haar functions [1]

$$\chi_0^{(0)}(x) \equiv 1 \quad (x \in [0, 1]);$$

$$\chi_m^{(k)}(x) = \begin{cases} \sqrt{2^m} & \left(x \in \left(\frac{k}{2^m}, \frac{2k+1}{2^{m+1}}\right)\right), \\ -\sqrt{2^m} & \left(x \in \left(\frac{2k+1}{2^{m+1}}, \frac{k+1}{2^m}\right)\right), \\ 0 & \left(x \in [0, 1] \setminus \left\{\left(\frac{k}{2^m}, \frac{2k+1}{2^{m+1}}\right) \cup \left(\frac{2k+1}{2^{m+1}}, \frac{k+1}{2^m}\right)\right\}\right), \end{cases}$$

($m = 0, 1, 2, \dots$; $k = 0, 1, 2, \dots, 2^m - 1$) and arrange them in a sequence $\{\chi_n\}_{n=0}^{\infty}$ by setting $\chi_0 = \chi_0^{(0)}$ and $\chi_n = \chi_m^{(k)}$ for $n = 2^m + k$.

The system of the functions

(1) $$\varphi_n(x) = \int_0^x \chi_n(t)\, dt \qquad (0 \leqq x \leqq 1;\ n = 0, 1, 2, \dots)$$

will be called the Schauder—Ciesielski system (cf. [2], [3]).

We associate with every function $f(x)$ on the interval [0, 1] the numerical sequence $\{c_n(f)\}_{n=0}^{\infty}$ defined by

(2) $$c_0(f) = f(1) - f(0),$$

$$c_n(f) = -\int_0^1 f(x)\, d\chi_n(x) \left(= \sqrt{2^m} \left[2f\left(\frac{2k+1}{2^{m+1}}\right) - f\left(\frac{k}{2^m}\right) - f\left(\frac{k+1}{2^m}\right) \right] \right)$$

$$(n = 2^m + k;\ m = 0, 1, 2, \dots;\ k = 0, 1, 2, \dots, 2^m - 1)$$

and call

(3) $$S(f) = f(0) + \sum_{n=0}^{\infty} c_n(f) \varphi_n(x)$$

the Schauder—Ciesielski series of the function $f(x)$.

If $f(x)$ is continuous on the interval [0, 1] then it is known that the series (3) converges uniformly to $f(x)$ ([2], [3]).

Theorem 1. *For an arbitrary function $f(x)$ on* [0, 1] *the series* (3) *converges to $f(x)$ at every dyadic rational point x, and at every point x where $f(x)$ is continuous.*

Proof. For the partial sums

$$S_n(x) = S_n(f;x) = f(0) + \sum_{l=0}^{n-1} c_l(f)\varphi_l(x) \tag{4}$$

we have evidently that

$$S_n(x) = S_{2^\nu}(x) \quad \text{or} \quad S_n(x) = S_{2^{\nu+1}}(x) \quad \text{if} \quad 2^\nu \leqq n < 2^{\nu+1}.$$

Thus it is enough to study the convergence behaviour of the partial sums $S_{2^\nu}(x)$.

Let x be an arbitrary but fixed point in the interval [0, 1]. For every ν we define the pair of dyadic rationals $\alpha_\nu(x)$, $\beta_\nu(x)$ by the inequalities:

$$\alpha_\nu(x) = \frac{l}{2^\nu} \leqq x < \frac{l+1}{2^\nu} = \beta_\nu(x) \qquad (0 \leqq l < 2^\nu - 1). \tag{5}$$

From (2) and (4) we deduce

$$S_{2^\nu}\left(f; \frac{l}{2^\nu}\right) = f\left(\frac{l}{2^\nu}\right) \qquad (\nu = 0, 1, 2, \ldots;\ l = 0, 1, \ldots, 2^\nu - 1). \tag{6}$$

Since $S_{2^\nu}(x)$ is linear on the interval $(\alpha_\nu(x), \beta_\nu(x))$, (2) and (4) imply that

$$S_{2^\nu}(f;x) = f(\alpha_\nu(x)) + 2^\nu(x - \alpha_\nu(x))\{f(\beta_\nu(x)) - f(\alpha_\nu(x))\}. \tag{7}$$

For dyadic irrational x our assertion follows from (7) if we take into account the inequalities $0 < (x - \alpha_\nu(x))2^\nu < 1$. For dyadic rational x the assertion is trivial.

2. A function $f(x)$ $(x \in [0, 1])$ will be called *D-differentiable* at x if

$$2^\nu[f(\beta_\nu(x)) - f(\alpha_\nu(x))]$$

tends to a finite limit $f'_D(x)$ as $\nu \to \infty$.

Theorem 2. *Let $f(x)$ be an arbitrary function defined on* [0, 1]. *Then the Haar series*

$$S'(x) = \sum_{n=0}^{\infty} c_n(f)\chi_n(x) \tag{8}$$

with the coefficients defined by (2) *is convergent at a dyadic irrational point x if and only if $f'_D(x)$ exists, and in this case its sum is equal to $f'_D(x)$.*

Proof. From (1) and (6) we deduce that

$$s_{2^\nu}(x) = \sum_{l=0}^{2^\nu-1} c_l(f)\chi_l(x) = \sum_{l=0}^{2^\nu-1} c_l(f)\varphi_l'(x) = S_{2^\nu}'(x) = \tag{9}$$

$$= 2^\nu[S_{2^\nu}(f;\beta_\nu(x)) - S_{2^\nu}(f;\alpha_\nu(x))] = 2^\nu[f(\beta_\nu(x)) - f(\alpha_\nu(x))].$$

Since

$$s_n(x) = s_{2^\nu}(x) \quad \text{or} \quad s_n(x) = s_{2^{\nu+1}}(x) \quad \text{for} \quad 2^\nu \leqq n < 2^{\nu+1}, \tag{10}$$

our assertion is an immediate consequence of (9).

Theorem 3. *The series*

$$\sum_{n=0}^{\infty} a_n\chi_n(x) \tag{11}$$

is the Haar—Fourier expansion of a Lebesgue-integrable function if and only if there exists an absolutely continuous function $F(x)$ for which $a_n = c_n(F)$, i.e.

$$S'(F) = \sum_{n=0}^{\infty} a_n\chi_n(x). \tag{12}$$

Proof. Suppose (11) is the Haar—Fourier expansion of a function $f(x)\in L[0,1]$, i.e. that

$$a_n = \int_0^1 f(x)\chi_n(x)\,dx \qquad (n = 0, 1, 2, \dots). \tag{13}$$

Let $F(x) = \int_0^x f(t)\,dt$. Integration by parts gives

$$a_n = \int_0^1 F'(x)\chi_n(x)\,dx = -\int_0^1 F(x)\,d\chi_n(x) = c_n(F) \tag{14}$$

for $n=1,2,3,\dots$, while for $n=0$ the equality $a_0 = c_0(F)$ is trivial.

Conversely, if $F(x)$ is absolutely continuous, then we can consider the series (11), where

$$a_n = -\int_0^1 F(x)\,d\chi_n(x) = \int_0^1 F'(x)\chi_n(x)\,dx,$$

i.e. $S'(F)$ is the Haar—Fourier expansion of the integrable function $F'(x)$.

Theorem 4. *All the partial sums of a Haar series* (11) *are non-negative everywhere if and only if there exists an increasing function $f(x)$ on* [0, 1] *for which $S'(F) = \Sigma a_n\chi_n(x)$.*

Proof. If $f(x)$ is an increasing function in [0, 1], then by the equation (9) the partial sums of the Haar series $S'(f)$ are non-negative everywhere.

Conversely, if for all $x \in [0, 1]$ and for every n the inequalities

$$s_n(x) = \sum_{l=0}^{n-1} a_l \chi_l(x) \geqq 0$$

hold, then the functions

$$S_n(x) = \int_0^x s_n(t)\,dt \qquad (n = 1, 2, \ldots) \tag{15}$$

are absolutely continuous and increasing for every n, and the equalities

$$S_n(0) = 0 \qquad S_n(1) = a_0$$

hold. Applying the theorem of Helly we can pick out a subsequence $\{S_{n_k}(x)\}_{k=1}^{\infty}$ of the sequence (15) such that the limit

$$\lim_{k\to\infty} S_{n_k}(x) = F(x) \tag{16}$$

exists for every x [4]. The function $F(x)$ is evidently increasing on [0, 1] and we have $F(0) = 0$ and $F(1) = a_0$.

Since the functions $S_{n_k}(x)$ satisfy the inequalities $0 \leqq S_{n_k}(x) \leqq a_0$ we have by the dominated convergence theorem of Lebesgue that

$$\lim_{k\to\infty} \int_0^1 S_{n_k}(x)\,d\chi_m(x) = \int_0^1 F(x)\,d\chi_m(x). \tag{17}$$

Since by the orthonormality of the Haar system the equalities

$$\int_0^1 S_{n_k}(x)\,d\chi_m(x) = -\int_0^1 \chi_m(x)\,dS_{n_k}(x) = -\int_0^1 \chi_m(x) s_{n_k}(x)\,dx = -a_m$$

are true for $n_k \geqq m$, it follows from (17) that

$$a_m = -\int_0^1 F\,d\chi_m, \quad \text{i.e.} \quad a_m = c_m(F) \qquad (m = 0, 1, 2, \ldots).$$

Q.E.D.

The following assertion can be proved in a quite similar way. (We have only to use the decomposition of a function $f(x)$ into its positive and negative parts.)

Theorem 5. *For the partial sums* $s_n(x) = \sum_{l=0}^{n-1} a_l \chi_l(x)$ *of a Haar series* (11) *we have*

$$I_n = \int_0^1 |s_n(x)|\,dx = O(1)$$

if and only if there exists a function $F(x)$ *of bounded variation on* [0, 1] *for which* $S'(F) = \sum_{l=0}^{\infty} a_l \chi_l(x)$.

3. In this section we mention some corollaries to the above theorems.

Theorem 6. *A Haar series with non-negative partial sums converges almost everywhere to* 0 *if and only if there exists an increasing singular function* $f(x)$ *for which the Haar series is identically equal to* $S'(f)$.

The theorem is an immediate consequence of Theorems 2 and 4.

Theorem 7. *There exists a Haar series with non-negative partial sums, which is not the Haar—Fourier expansion of some function* $f(x)\in L[0, 1]$.

This follows from Theorem 6 if we apply it to an increasing singular function. Thus the fact, conjectured by Steinhaus for trigonometric series, and proved in [5], does not hold for Haar series.

Finally it is possible to give a class of continuous functions which are not differentiable almost everywhere; more precisely the following assertion is true:

Theorem 8. *If in a Schauder—Ciesielski series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{18}$$

the coefficients a_n *tend monotonically to* 0 *then the series tends uniformly to a continuous function* $f(x)$ *and if at the same time*

$$\sum a_n^2 = \infty, \tag{19}$$

then $f(x)$ *is not differentiable almost everywhere.*

Proof. The series (18) is equiconvergent with the series

$$\sum_{m=0}^{\infty}\left(\sum_{k=0}^{2^m-1} a_{2^m+k}\varphi_{2^m+k}(x)\right) \equiv \sum_{m=0}^{\infty} \Phi_m(x),$$

and since $|\Phi_m(x)| \leqq a_0 2^{-\frac{m}{2}}$, the series (18) converges uniformly, and hence it is the Schauder—Ciesielski expansion of its continuous sum $f(x)$.

If the function $f(x)$ were differentiable on a set $E\subset[0, 1]$ of positive measure, then by Theorem 2 the Haar series

$$S'(f) = \sum_{n=0}^{\infty} a_n \chi_n(x)$$

would be convergent on E. But this is impossible, because, by [6], a Haar series is almost everywhere divergent if its coefficients are monotonic and satisfy condition (19).

References

[1] A. HAAR, Zur Theorie der orthogonalen Funktionensysteme, *Dissertation* (Göttingen, 1909) und *Math. Annalen,* **69** (1910), 331—371, und **71** (1912), 33—53.

[2] S. KACZMARZ und H. STEINHAUS, *Theorie der Orthogonalreihen* (Warszawa—Lwów, 1935).

[3] Z. CIESIELSKI, On Haar functions and on the Schauder basis of the space $C\langle 0, 1\rangle$, *Bull. Acad Polon. Sci.,* **7** (1959), 227—232.

[4] A. ZYGMUND, *Trigonometric series.* I (Cambridge, 1959), p. 137.

[5] Y. KATZNELSON, Trigonometric series with positive partial sums, *Bull. Amer. Math. Soc.,* **71** (1965), 718—719.

[6] П. Л. Ульянов, О рядах по системе Хаара с монотонными коэффициентами, *Изв. Акад. Наук СССР,* **28** (1964), 925—950.

(Received August 9, 1968)