Operator conjugation with respect to symmetric and skew-symmetric forms

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Let K be the real or complex number field, E the n-dimensional $(1 \leqq n < \infty)$ vector space over K, and B the set of all linear mappings of E into itself. The elements of B will be termed *operators*.

We recall that a *sesquilinear form* on E is a function $\varphi(x, y)$ of two variables $x, y \in E$ with values in K such that

$$\varphi(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 \varphi(x_1, y) + \alpha_2 \varphi(x_2, y)$$

and

$$\varphi(y, \alpha_1 x_1 + \alpha_2 x_2) = \bar{\alpha}_1 \varphi(y, x_1) + \bar{\alpha}_2 \varphi(y, x_2)$$

for every $\alpha_1, \alpha_2 \in K$; $x_1, x_2, y \in E$. The sesquilinear form φ is said to be *non-degenerate* if $\varphi(x, y) = 0$ for every y implies $x = 0$, and $\varphi(x, y) = 0$ for every x implies $y = 0$. The sesquilinear form φ is said to be *symmetric* if $\varphi(x, y) = \overline{\varphi(y, x)}$, and *skew-symmetric* if $\varphi(x, y) = -\overline{\varphi(y, x)}$ for every x, y in E.

Let $\varphi(x, y)$ be a symmetric or skew-symmetric non-degenerate sesquilinear form on E. It is well-known (see e.g. [1], section 99) that to every $T \in B$ there is a uniquely defined $T^* \in B$ such that

$$\varphi(Tx, y) = \varphi(x, T^* y) \qquad (x, y \in E;\ T \in B). \tag{1}$$

The operator T^* is called the *adjoint* of T with respect to φ. The mapping $T \to T^*$ $(T \in B)$ has the following properties:

$$(\alpha_1 T_1 + \alpha_2 T_2)^* = \bar{\alpha}_1 T_1^* + \bar{\alpha}_2 T_2^* \quad (\alpha_1, \alpha_2 \in K;\ T_1, T_2 \in B), \tag{2}$$

$$(T_1 T_2)^* = T_2^* T_1^* \quad (T_1, T_2 \in B), \tag{3}$$

$$T^{**} = T \quad (T \in B). \tag{4}$$

In the present note it will be shown that any mapping of B into itself satisfying the conditions (2)—(4) can be obtained in this way.

The idea was suggested us by E. FRIED's paper [2] where the case of a symmetric $\varphi(x, y)$ with *definite* $\varphi(x, x)$ (i.e. $\varphi(x, x) = 0$ only if $x = 0$) is treated.

The "existence" part of our result is implicitly contained in the concluding remarks of [2]. However, the question of uniqueness and the separate characterization of the symmetric and skew-symmetric case do not appear there. Our proof of existence seems to be not quite different from FRIED's one, but we need neither an *a priori* given inner product nor the characterization of operators commuting with every element of B.

Theorem. *Let $T \to T^*$ $(T \in B)$ be a mapping of B into itself with the properties* (2), (3), (4). *If $K=C$, the field of complex numbers, then there exist both a symmetric and a skew-symmetric non-degenerate sesquilinear form φ on E satisfying the relation* (1). *If $K=R$, the field of real numbers, then there exists either a symmetric or a skew-symmetric non-degenerate sesquilinear form φ on E satisfying* (1); *the form φ will be symmetric if and only if there is an operator $T_0 \in B$ such that*

$$\text{rank } T_0 = 1, \qquad T_0^* T_0 \neq 0. \tag{5}$$

In each of the above cases φ is, up to a real non-zero factor, uniquely determined.

Proof. If there is an operator T_0 with properties (5), we choose a basis $e_1, \ldots, e_n$ of E such that

$$T_0 e_k = 0 \qquad (k=2, \ldots, n). \tag{6}$$

In the opposite case let $e_1, \ldots, e_n$ be any basis of E.

We define n^2 operators P_{jk} setting

$$P_{jk} e_r = \delta_{kr} e_j \qquad (j, k, r = 1, \ldots, n). \tag{7}$$

Turning our attention to the mapping $T \to T^*$ $(T \in B)$ we observe that

$$T^* = 0 \quad \text{if and only if} \quad T = 0. \tag{8}$$

Indeed, according to (2), $0^* = (2 \cdot 0)^* = 2 \cdot 0^*$ i.e. $0^* = 0$. Moreover, in view of (4), $T^* = 0$ implies $T = T^{**} = (T^*)^* = 0^* = 0$.

Since by the definition (7)

$$P_{jk} \neq 0, \quad P_{jk} P_{rs} = \delta_{kr} P_{js} \quad (j, k, r, s = 1, \ldots, n), \tag{9}$$

from (8) and (3) we infer

$$P_{jk}^* \neq 0, \quad P_{rs}^* P_{jk}^* = \delta_{kr} P_{js}^* \quad (j, k, r, s = 1, \ldots, n). \tag{10}$$

We will construct and study the required sesquilinear form φ by the aid of the basis $e_1, \ldots, e_n$ and a new basis $f_1, \ldots, f_n$. In order to define the latter, we need the operators P_{jk}^* and an operator S (resp. W) of rank 1 such that $S^* = S$ $(W^* = -W)$.

Let the relations (5) be satisfied by some $T_0 \in B$. We may assume that (6) holds, too. Setting $S = T_0^* T_0$ we have

$$S^* = S, \quad S \neq 0, \quad S e_k = 0 \qquad (k=2, \ldots, n). \tag{11}$$

If (5) cannot be satisfied then in particular

$$P_{11}^* P_{11} = 0, \quad \text{and} \quad P_{k1}^* P_{k1} = 0, \quad (P_{11} + P_{k1})^*(P_{11} + P_{k1}) = 0 \qquad (k = 2, \ldots, n).$$

Hence, in view of (2),

$$P_{11}^* P_{k1} + P_{k1}^* P_{11} = 0 \qquad (k = 2, \ldots, n).$$

But for some $k \neq 1$ the operator $W = P_{11}^* P_{k1}$ is different from zero. In fact, otherwise it would follow that

$$P_{11}^* e_k = P_{11}^* P_{k1} e_1 = 0 \qquad (k = 1, \ldots, n)$$

i.e. $P_{11}^* = 0$. Thus we have

$$W^* = -W, \quad W \neq 0, \quad We_k = 0 \qquad (k = 2, \ldots, n). \tag{12}$$

Let

$$f_1 = \begin{cases} Se_1 & \text{if (5) can be fulfilled,} \\ We_1 & \text{otherwise,} \end{cases} \tag{13}$$

and let

$$f_k = P_{1k}^* f_1 \qquad (k = 2, \ldots, n). \tag{14}$$

The validity of (14) can be extended to $k = 1$. Really, if $f_1 = Se_1$, then $P_{11}^* f_1 = P_{11}^* Se_1 = P_{11}^* S^* e_1 = (SP_{11})^* e_1 = S^* e_1 = Se_1 = f_1$; in the case $f_1 = We_1$ a similar argument holds. Making use of this fact and of formula (10) we obtain:

$$P_{jk}^* f_r = \delta_{jr} f_k \qquad (j, k, r = 1, \ldots, n). \tag{15}$$

Assume that $\sum_{r=1}^{n} \alpha_r f_r = 0$ for some $\alpha_1, \ldots, \alpha_n \in K$. Then $P_{j1}^* \sum_{r=1}^{n} \alpha_r f_r = 0$, so that by (15) $\alpha_j f_1 = 0$ $(j = 1, \ldots, n)$. On account of (11)—(13) $f_1 \neq 0$. Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Consequently, $f_1, \ldots, f_n$ constitute a basis of E.

Now let $\varphi(x, y)$ be a non-degenerate sesquilinear form on E which satisfies the relation (1). Then necessarily

$$\varphi(e_j, f_k) = \varphi(P_{j1} e_1, f_k) = \varphi(e_1, P_{j1}^* f_k) = \delta_{jk} \varphi(e_1, f_1) \qquad (j, k = 1, \ldots, n),$$

and

$$\varphi\left(\sum_{j=1}^{n} \mu_j e_j, \sum_{k=1}^{n} \nu_k f_k\right) = \sum_{j,k=1}^{n} \mu_j \bar{\nu}_k \varphi(e_j, f_k) =$$

$$= \sum_{k=1}^{n} \mu_k \bar{\nu}_k \varphi(e_1, f_1) \qquad (\mu_k, \nu_k \in K;\ k = 1, \ldots, n),$$

where

$$\varphi(e_1, f_1) \neq 0.$$

Therefore if φ_1, φ_2 are two non-degenerate sesquilinear forms with the property

$$\varphi_k(Tx, y) = \varphi_k(x, T^*y) \qquad (x, y \in E;\ T \in B;\ k = 1, 2),$$

then $\varphi_1 = \lambda\varphi_2$, where

$$\lambda = \frac{\varphi_1(e_1, f_1)}{\varphi_2(e_1, f_1)} \tag{16}$$

is a non-zero (till now possibly complex) number.

Since a real multiple of a symmetric (skew-symmetric) form is symmetric (skew-symmetric), and a non-degenerate form cannot be symmetric and skew-symmetric at the same time, for $K = R$ we additionally find that the cases where φ can be chosen. symmetric or skew-symmetric, respectively, must be mutually disjoint.

If $K = C$ and both of the forms φ_1, φ_2 are required to be symmetric (resp. skew-symmetric), then the value of λ in (16) must be real. As a matter of fact, the relations

$$\varphi_k(e_1, f_1) = \varepsilon\overline{\varphi_k(f_1, e_1)} \qquad (\varepsilon = \pm 1;\ k = 1, 2)$$

imply $\lambda = \bar{\lambda}$.

Conversely, it is easy to see that for any fixed real non-zero number γ the formula

$$\varphi\left(\sum_{j=1}^{n} \mu_j e_j, \sum_{k=1}^{n} \nu_k f_k\right) = \gamma \sum_{k=1}^{n} \mu_k \bar{\nu}_k \qquad (\mu_k, \nu_k \in K;\ k = 1, \dots, n) \tag{17}$$

defines a non-degenerate sesquilinear form on E. Moreover, in view of (7) and (15) we have

$$\varphi(P_{jk}e_r, f_s) = \delta_{kr}\,\varphi(e_j, f_s) = \delta_{kr}\delta_{js}\gamma,$$

$$\varphi(e_r, P_{jk}^* f_s) = \delta_{js}\varphi(e_r, f_k) = \delta_{js}\delta_{kr}\gamma$$

i.e.

$$\varphi(P_{jk}e_r, f_s) = \varphi(e_r, P_{jk}^* f_s) \qquad (j, k, r, s = 1, \dots, n),$$

so that making use of the linearity of P_{jk} and the sequilinearity of φ we obtain

$$\varphi(P_{jk}x, y) = \varphi(x, P_{jk}^* y) \qquad (x, y \in E;\ j, k = 1, \dots, n).$$

Taking into account that any $T \in B$ is a linear combination of the operators P_{jk}, the relation (1) follows.

If $f_1 = Se_1$ (cf. (13)), then the relations (1), (11), (17), (15), (4) and (7) yield:

$$\varphi(f_1, e_1) = \varphi(Se_1, e_1) = \varphi(e_1, Se_1) = \varphi(e_1, f_1) = \gamma,$$

$$\varphi(f_k, e_j) = \varphi(P_{1k}^* f_1, e_j) = \varphi(f_1, P_{1k}e_j) = \delta_{kj}\varphi(f_1, e_1) = \delta_{kj}\gamma,$$

$$\varphi\left(\sum_{k=1}^{n} \nu_k f_k, \sum_{j=1}^{n} \mu_j e_j\right) = \sum_{j,k=1}^{n} \nu_k \bar{\mu}_j \varphi(f_k, e_j) = \sum_{k=1}^{n} \nu_k \bar{\mu}_k \gamma.$$

Thus, in view of (17), φ is symmetric.

If $f_1 = We_1$ (cf. (13)) then, by virtue of (12), φ turns out to be skew-symmetric.

Finally, let $K = C$, and let φ be a symmetric (skew-symmetric) non-degenerate sesquilinear form satisfying the relation (1). Then $\varphi_1 = i\varphi$ is a skew-symmetric (resp. symmetric) non-degenerate sesquilinear form satisfying (1).

Added in proof. Professor KLAUS VALA kindly called my attention to the fact that FRIED's result referred to in the introduction is a special case of a theorem of Mackey and Kakutani (cf. C. E. RICKART, *General theory of Banach algebras*, Princeton—Toronto—London—New York, 1960; p. 265), where operators on a Banach space of arbitrary dimension are considered. The two proofs, however, seem to have nothing in common.

References

[1] А. И. Мальцев, *Основы линейной алгебры* (Москва, 1956).

[2] E. FRIED, A characterization of the adjoints of linear transformations, *Annales Univ. Sci. Budapest*, sectio math., **8** (1965), 181—185.

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