J-unitary dilation of a general operator

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According to a well-known theorem of SZ.-NAGY [13], [14, Thm. I. 4. 2], every contraction on Hilbert space has a unitary dilation on a larger space (and also has an extension on a larger space which is the adjoint of an isometry [14, Thm. I. 4. 1.). In this paper, by a modification of familiar methods, the corresponding result is obtained for an arbitrary closed densely-defined operator. The conclusion is different in that the dilation is only J-unitary (and the extension is only the adjoint of a J-isometry, or the J-adjoint of a J-isometry).

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1. Definitions

The subject will be a closed operator T whose domain $\boldsymbol{D}(T)$ is a dense linear set in a Hilbert space $\mathfrak{H}$. The inner product of $\mathfrak{H}$ will be denoted by $(\,,)$. I will construct later a Hilbert space $\mathfrak{K}$ of which $\mathfrak{H}$ is a linear subspace; the inner product of $\mathfrak{K}$ will be an extension of that of $\mathfrak{H}$, and will also be denoted by $(\,,)$. The orthoprojector on $\mathfrak{K}$ onto $\mathfrak{H}$ will be denoted by $P_{\mathfrak{H}}$. I will also construct an operator U, closed and densely defined in $\mathfrak{K}$, which is a "dilation" of T; this means that

$$(1.1) \qquad T^n = P_{\mathfrak{H}} U^n|_{\mathfrak{H}} \quad \text{and} \quad T^{*n} = P_{\mathfrak{H}} U^{-n}|_{\mathfrak{H}} \qquad (n = 1, 2, \ldots).$$

In addition $\mathfrak{K}$ will be a "J-space". This means [7] that the Hilbert space $\mathfrak{K}$ will have associated with it a canonical symmetry J, i.e., a fixed unitary hermitian operator J. In any J-space one considers along with J the complementary orthoprojectors J^+ and J^-. (I will use the notations A^+ and A^- for the positive part and the negative part of an arbitrary self-adjoint operator A [11, § 108].) It is often assumed that the ranges $\boldsymbol{R}(J^+)$ and $\boldsymbol{R}(J^-)$ are both non-zero, but here that is not

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assumed. In terms of J, a new continuous hermitian sesquilinear form is defined by

$$[k, k'] = (Jk, k') \qquad (k, k' \in \mathfrak{K}).$$

Unlike the inner product $(,)$, the "J-product" $[,]$ need not be definite; in particular, $[k, k] > 0$ for non-zero $k \in \boldsymbol{R}(J^+)$, while $[k, k] < 0$ for non-zero $k \in \boldsymbol{R}(J^-)$. For this reason J-spaces are also called Hilbert spaces with indefinite metric; but do not be misled. The norm is defined in terms of the definite inner product, not the J-product, and topological notions are defined in terms of the norm. The "J-adjoint" of any A is JA^*J.

A "J-isometric" operator U is a closed, densely defined operator which preserves the J-product:

(1.2)
$$[Uk, Uk'] = [k, k'] \qquad (k, k' \in \boldsymbol{D}(U)).$$

A J-isometry U is called "J-unitary" in case it has a densely defined inverse, which then is necessarily J-isometric as well. The terminology and notation of unbounded operators are used because the operator U which appears below really can be unbounded. This has obliged me to depart from the usual terminology [7], in which J-unitary operators are by definition bounded. (Iohvidov [5], [6] studies unbounded J-isometries, but in quite different context.)

2. The main lemma

Let $|T|$ denote $(T^*T)^{1/2}$, a self-adjoint operator with $\boldsymbol{D}(|T|) = \boldsymbol{D}(T)$, $\boldsymbol{R}(|T|) = \boldsymbol{R}(T^*)$; similarly $|T^*| = (TT^*)^{1/2}$. Let W denote the unique partial isometry such that $T = W|T| = |T^*|W$, $\boldsymbol{R}(W) = \overline{\boldsymbol{R}(T)}$, $\boldsymbol{R}(W^*) = \overline{\boldsymbol{R}(T^*)}$ [10].

It will be useful to have special notation and terminology for some operators and subspaces derived from these, which will figure prominently in the construction. Let $|T| = \int_0^\infty \lambda \, dE(\lambda)$ be the spectral resolution of $|T|$; $|T^*| = \int_0^\infty \lambda \, dF(\lambda)$, that of $|T^*|$. Define

$$J_T = \operatorname{sgn}(1 - T^*T) = \int_0^\infty \operatorname{sgn}(1 - \lambda^2) \, dE(\lambda),$$

$$Q_T = (|1 - T^*T|)^{1/2} = \int_0^\infty \sqrt{|1 - \lambda^2|} \, dE(\lambda),$$

$$D_T = ((1 - T^*T)^+)^{1/2} = \int_0^1 \sqrt{1 - \lambda^2} \, dE(\lambda),$$

$$X_T = ((1 - T^*T)^-)^{1/2} = \int_1^\infty \sqrt{\lambda^2 - 1} \, dE(\lambda),$$

which are self-adjoint operators. Clearly J_T and D_T are bounded and everywhere defined. As for the possibly unbounded operators $|T|$, Q_T, and X_T, they differ by operators which are bounded, and so they have the same domain (which is $\boldsymbol{D}(T)$).

In case T is a contraction, J_T^- and X_T are O, while Q_T is equal to D_T, the same operator denoted by that symbol in [14].

In spite of the choice of letter, J_T is not quite suitable for defining a J-product on $\mathfrak{H}$, since it can have a null-space. Its role in the eventual construction of the J-space $\mathfrak{K}$ will be less direct.

It is obvious that $J_T Q_T^2 = Q_T^2 J_T = 1 - T^* T$. It is less immediate, but worth noting, that

$$(2.1)\qquad (J_T Q_T h, Q_T l) = (h, l) - (Th, Tl) \qquad (h, l \in \boldsymbol{D}(T)).$$

To see this, we may introduce $|||\ |||$, the "graph norm for T", defined by

$$|||h|||^2 = \|h\|^2 + \|Th\|^2 \qquad (h \in \boldsymbol{D}(T)).$$

It makes $\boldsymbol{D}(T)$ into a (complete) Hilbert space, in which it is easy to prove that $\boldsymbol{D}(T^* T)$ is dense. But with respect to this norm, both sides of (2. 1) are continuous functions of h and l; and (2. 1) does hold for h and l belonging to $\boldsymbol{D}(T^* T)$; therefore it must hold in general.

Interchanging T with T^* and $E(\ )$ with $F(\ )$, we get operators J_{T^*}, Q_{T^*}, D_{T^*}, X_{T^*}, with properties corresponding.

Lemma. *Let the symbol* $\blacksquare$ *stand for either J, Q, D, or X. We have* $T\blacksquare_T = \blacksquare_{T^*} T$.

In the case $TD_T = D_{T^*} T$, this relation has been crucial in unitary dilation theory since its beginnings [5], and it continues its role here.

Proof. Each relation to be proved has the form

$$(2.2)\qquad W \int_0^\infty f(\lambda)\, dE(\lambda) = \left(\int_0^\infty f(\lambda)\, dF(\lambda)\right) W$$

for some piecewise-continuous function f vanishing at 0: for example, in the case of the equation $TX_T = X_{T^*} T$, take $f(\lambda) = \chi_{[1,\infty]}(\lambda) \cdot \lambda \sqrt{\lambda^2 - 1}$. Now the fact that $W|T| = |T^*| W$, with which we began, implies that $WE(\lambda) = F(\lambda) W$ for all λ if $E(\ )$ and $F(\ )$ are normalized in the same way. Using the criterion $\int_0^\infty f(\lambda)^2\, d(E(\lambda)h, h) < \infty$ for a vector h to belong to the domain of $\int_0^\infty f(\lambda)\, dE(\lambda)$ and using the properties of W, we see easily that the two sides of (2. 2) have the same domain, and agree on the domain.

This proves the Lemma as stated, but it is worth noting the somewhat more delicate fact that (with the same notation)

$$(\blacksquare_T h, T^* l) = (Th, \blacksquare_{T^*} l) \qquad (h \in \boldsymbol{D}(T), l \in \boldsymbol{D}(T^*)). \tag{2. 3}$$

For J_T and D_T this does follow from the Lemma. For the other two cases, we may either use approximation in the graph norm as for (2. 1), or else use the fact that Q_T resp. X_T differs from $|T|$ by a bounded operator, which reduces (2. 3) rather quickly to the Lemma.

The operators J_T, J_{T^*}, Q_T, Q_{T^*} are all that are needed to prove the main theorem; the others will be used only for the discussion of the geometry of the dilation space, which will follow in § 4.

3. The dilation

Theorem. *Given any closed, densely defined operator T in $\mathfrak{H}$, there exists a Hilbert space $\mathfrak{K} \supseteqq \mathfrak{H}$ and there exists a closed, densely defined operator U in $\mathfrak{K}$, with the following properties:*

(a) *$\mathfrak{K}$ is a J-space, with $\mathfrak{H} \subseteqq J^+(\mathfrak{K})$* (i.e., $[h, l] = (h, l)$ for $h, l \in \mathfrak{H}$);

(b) *U is J-unitary, that is,* (1. 2) *holds and U^{-1} is densely defined;*

(c) *U is a dilation of T, that is,* (1. 1) *holds;*

(d) $\bigvee\{U^n \mathfrak{H}\colon n = 0, \pm 1, \pm 2, \dots\} = \mathfrak{K}$.

(In stating (d), and occasionally below, I use an expression like $U\mathfrak{H}$ as short-hand for $U(\mathfrak{H} \cap \boldsymbol{D}(U))$.)

The construction follows quite closely that of Schäffer, as sharpened subsequently [14, I. 5]. I begin (as Schäffer did) with a space $\mathfrak{K}^0$, somewhat larger than desired but easy to describe: it is the direct sum of countably many isometric copies of $\mathfrak{H}$ (with the usual inner product). One of these copies I will identify at once with $\mathfrak{H}$. The unitary application of each copy onto the next one in order will be denoted by S; thus I write

$$\mathfrak{K}^0 = \dots \oplus S^{-2}\mathfrak{H} \oplus S^{-1}\mathfrak{H} \oplus \mathfrak{H} \oplus S\mathfrak{H} \oplus S^2\mathfrak{H} \oplus \cdots. \tag{3. 1}$$

S will have a role as an operator (a bilateral shift of multiplicity $\dim \mathfrak{H}$ acting on $\mathfrak{K}$) and as a device for indexing the component subspaces of $\mathfrak{K}^0$ in (3. 1). For any $k \in \mathfrak{K}^0$, k_i will mean the component of k in $S^i\mathfrak{H}$. (Thus, for instance, $(Sk)_i = Sk_{i-1}$.)

Now to define J, I take the diagonal operator-matrix

$$
\left[\begin{array}{cc|c|cc}
\ddots & & & & \\
J_{T^*} & & & & \\
& J_{T^*} & & & \\
\hline
& & 1 & & \\
\hline
& & & J_T & \\
& & & & J_T \\
& & & & & \ddots
\end{array}\right]. \tag{3.2}
$$

That is, for an element $S^n h$ $(h \in \mathfrak{H})$ I define

$$
J(S^n h) = \begin{cases} S^n(J_{T^*}h) & (n < 0), \\ h & (n = 0), \\ S^n(J_T h) & (n > 0). \end{cases}
$$

This extends to a self-adjoint operator J on $\mathfrak{K}^0$ such that the space is spanned by its eigenspaces belonging to (at most) the three eigenvalues $0, \pm 1$. Define $\mathfrak{K} = \boldsymbol{R}(J) \subseteqq \mathfrak{K}^0$ and $[k, k'] = (Jk, k')$. Obviously conclusion (a) of the Theorem holds.

The operator U will be, essentially, an extension of

$$
V = \left[\begin{array}{cccc|c|cccc}
\ddots & & & & & & & & \\
& O & & & & & & & \\
& 1 & O & & & & & & \\
& & 1 & O & & & & & \\
\hline
& & & Q_{T^*} & T & & & & \\
\hline
& & & -J_T T^* & Q_T & O & & & \\
& & & & & 1 & O & & \\
& & & & & & 1 & O & \\
& & & & & & & & \ddots
\end{array}\right]. \tag{3.3}
$$

The simplest interpretation of (3. 3) is that it defines an operator in $\mathfrak{K}^0$ for which

$$
V(S^n h) = \begin{cases} S^{n+1}h & (n \neq -1, 0), \\ Q_{T^*}h \oplus -SJ_T T^* h \in \mathfrak{H} \oplus S\mathfrak{H} & (n = -1;\ h \in \boldsymbol{D}(T^*)), \\ Th \oplus SQ_T h \in \mathfrak{H} \oplus S\mathfrak{H} & (n = 0;\ h \in \boldsymbol{D}(T)), \end{cases}
$$

and which is then to be extended by linearity to the finite linear combinations of the vectors $S^n h$.

So defined, V takes $\boldsymbol{D}(V) \cap \mathfrak{K}$ into $\mathfrak{K}$. To verify this, we treat $(Vk)_1$, the only component about which there is any possible doubt. By definition, $(Vk)_1 = SQ_T k_0 - SJ_T T^* Sk_{-1}$. This is in $S(\boldsymbol{R}(J_T))$ as desired because Q_T and J_T, being by their definition self-adjoint operators with the same null-space, have $\overline{R(Q_T)} = \overline{R(J_T)}$ $(=R(J_T))$. Therefore we are free to treat (3. 3) as representing an operator with domain in $\mathfrak{K}$.

Next it must be proved that the operator V possesses a closure [10]. Suppose not; then there exists a sequence $(k^{(n)})$ in $D(V)$ such that $k^{(n)} \to 0$ but $Vk^{(n)}$ approaches a non-zero limit. It is easy to see that either $(Vk^{(n)})_0$ or $(Vk^{(n)})_1$ approaches a non-zero limit; the argument goes the same in either case, I will write it for the first alternative. We have $(Vk^{(n)})_0 = Tk_0^{(n)} + Q_{T^*}Sk_{-1}^{(n)}$ by definition, where $(k_0^{(n)})$ is a sequence of vectors in $D(T)$, tending to 0 as $n \to \infty$, and $(h'^{(n)}) = (Sk_{-1}^{(n)})$ is a sequence of vectors in $D(T^*)$, tending to 0 as $n \to \infty$; but where the sequence $((Vk^{(n)})_0)$ approaches a non-zero limit. We can write $Tk_0^{(n)} = |T^*|h^{(n)}$, where $(h^{(n)}) = (Wk_0^{(n)})$ is a sequence of elements of $D(T^*)$ approaching 0 as $n \to \infty$. Now

$$(Vk^{(n)})_0 = |T^*|h^{(n)} + Q_{T^*}h'^{(n)} = |T^*|(h^{(n)} + h'^{(n)}) + (Q_{T^*} - |T^*|)h'^{(n)},$$

and because $Q_{T^*} - |T^*|$ is bounded, the last term tends to 0 as $n \to \infty$. Therefore $|T^*|(h^{(n)} + h'^{(n)})$ tends to a non-zero limit. But this contradicts the fact that $|T^*|$ is a closed operator.

We may, therefore, extend V to a minimal closed linear operator U; its domain is clearly dense. It is also clear that $U(S^{-1}\mathfrak{H} \oplus \mathfrak{H}) \subseteq \mathfrak{H} \oplus S\mathfrak{H}$.

As to the first equation in (1. 1), clearly T^n is an extension of $P_{\mathfrak{H}}U^n|\mathfrak{H}$. It is also easy to see that $D(U^n) \cap \mathfrak{H} = D(T^n)$.

The other half of (1. 1) is less apparent, and will be deferred.

The main formal idea in the construction comes out in verifying that U is J-isometric, that is, in verifying (1. 2).

In doing this, we may restrict attention to vectors in $D(V)$ — thus to k such that $k_0 \in D(T)$ and $k_{-1} \in S^{-1}(D(T^*))$. This is by an argument already invoked in § 2: namely, if $D(U)$ is considered in the graph norm for U, then $D(V)$ is dense in it and both sides of (1. 2) are continuous in k and k'. Also, by the usual polarization argument, it is enough to prove (1. 2) for $k = k'$. Define l by

$$k = k_{-1} \oplus k_0 \oplus l;$$

this decomposition is both orthogonal and J-orthogonal. It is obvious that the transformation $l \to Ul = Sl$ preserves both the inner product and the J-product. It is also obvious that $U(k_{-1} \oplus k) \in \mathfrak{H} \oplus S\mathfrak{H}$ is both orthogonal and J-orthogonal to Ul. Therefore it is enough to prove (1. 2) for $l = 0$. The right-hand member is then

$$[k, k] = (J(k_{-1} \oplus k_0), k_{-1} \oplus k_0) = (J_{T^*}Sk_{-1}, Sk_{-1}) + (k_0, k_0).$$

The left-hand member of (1. 2) is

$$(3.4)\qquad [Uk, Uk] = (J((Uk)_0 \oplus (Uk)_1), (Uk)_0 \oplus (Uk)_1)$$
$$= ((Uk)_0, (Uk)_0) + (J_T S^{-1}(Uk)_1, S^{-1}(Uk)_1)$$
$$= (Q_{T^*}Sk_{-1} + Tk_0, Q_{T^*}Sk_{-1} + Tk_0) +$$
$$+ (J_T(-J_T T^* Sk_{-1} + Q_T k_0), -J_T T^* Sk_{-1} + Q_T k_0).$$

The terms $(Tk_0, Tk_0) + (J_T Q_T k_0, Q_T k_0)$ add to (k_0, k_0) by (2. 1). The terms

$$2\,\mathrm{Re}\,(Q_{T^*}Sk_{-1}, Tk_0) + 2\,\mathrm{Re}\,(-J_T^2 T^* Sk_{-1}, Q_T k_0) =$$
$$= 2\,\mathrm{Re}\,\{(Q_{T^*}Sk_{-1}, Tk_0) - (T^* Sk_{-1}, Q_T k_0)\}$$

add to zero essentially by the Lemma — strictly, by (2. 3). The remaining terms in (3. 4) are equal to

$$(Q_{T^*}Sk_{-1}, Q_{T^*}Sk_{-1}) + (T^* Sk_{-1}, J_T T^* Sk_{-1}).$$

I apply (2. 1) (with T and T^* interchanged) to this expression, taking as the vectors h, l in (2. 1) the vectors $J_{T^*} Sk_{-1}, Sk_{-1} \in \boldsymbol{D}(T^*)$. It becomes

$$(Q_{T^*}J_{T^*}h, Q_{T^*}l) + (T^* J_{T^*}h, J_T T^* l) = (J_{T^*}Q_{T^*}h, Q_{T^*}l) + (J_T T^* h, J_T T^* l) = (h, l),$$

by use also of the Lemma. That is,

$$[Uk, Uk] = (k_0, k_0) + (h, l) = (k_0, k_0) + (J_{T^*} Sk_{-1}, Sk_{-1}) = [k, k].$$

(1. 2) is established.

It is also easy to prove the "forward half" of (d). Define

$$(3.5)\qquad \mathfrak{K}_+ = \mathfrak{H} \oplus SJ_T(\mathfrak{H}) \oplus S^2 J_T(\mathfrak{H}) \oplus \cdots.$$

Since $U\mathfrak{H} \supseteqq SQ_T\mathfrak{H}$, which is dense in $SJ_T\mathfrak{H} = S\mathfrak{H} \cap \mathfrak{K}$, and since for $n > 1$ we have $U^n\mathfrak{H} = S^{n-1}U\mathfrak{H}$, it is clear (remembering (3.2)) that $\vee\{U^n\mathfrak{H}: n = 0, 1, 2, \dots\} = \mathfrak{K}_+$.

The remaining arguments concern the inverse of U. We know U is one-one by (1. 2); because if $Uk = 0$ then (1. 2) shows that Jk is orthogonal to the dense set $\boldsymbol{D}(U)$, and, J having zero null-space, this forces $k = 0$. Therefore U^{-1} exists; we have to consider $\boldsymbol{D}(U^{-1}) = \boldsymbol{R}(U)$.

Now it is evident that $\boldsymbol{R}(U) \supseteqq \oplus\{S^n\mathfrak{H}: n \neq 0,1\}$, and that when we consider restrictions to this subspace, U^{-1} agrees with S^{-1}. Take $k = k_0 \oplus k_1$, with $k_0 \in \boldsymbol{D}(T^*)$ and $k_1 \in S(\boldsymbol{D}(T))$. Define

$$(3.6)\qquad l = l_{-1} \oplus l_0 \quad (Sl_{-1} = J_{T^*}Q_{T^*}k_0 - J_{T^*}TS^{-1}k_1, \quad l_0 = T^*k_0 + Q_T J_T S^{-1} k_1).$$

I will show that $Ul = k$. Circumspection is needed with this l because it need not be in $\boldsymbol{D}(V)$. As in earlier arguments, let us approximate by elements from the appropriate domains. Let $(k^{(n)})$ be a sequence of elements of $\mathfrak{K}$ such that (i) $k^{(n)} = k_0^{(n)} \oplus k_1^{(n)}$; (ii) $(k_0^{(n)})$ is a sequence approaching k_0 in $\boldsymbol{D}(T^*)$ considered with the graph norm for T^*, while each $k_0^{(n)}$ lies in the dense set $\boldsymbol{D}(TT^*)$; (iii) similarly, in $\boldsymbol{D}(T)$ with the graph norm for T, $S^{-1}k_1^{(n)} \to S^{-1}k_1$ and $S^{-1}k_1^{(n)} \in \boldsymbol{D}(T^*T)$. In particular,

$k^{(n)} \to k$ in $\mathfrak{K}^0$. Now define $l^{(n)}$ in terms of $k^{(n)}$ by putting superscripts on equations (3. 6). By the definition of the graph norm we have also $l^{(n)} \to l$ in $\mathfrak{K}^0$. Furthermore $l^{(n)} \in \mathfrak{K}$, $l \in \mathfrak{K}$ by the definition of $\mathfrak{K}$. But $Ul^{(n)} = Vl^{(n)}$ can be computed from (3. 3). One obtains, using the Lemma,

$$(Vl^{(n)})_0 = Q_{T^*}(J_{T^*} Q_{T^*} k_0^{(n)} - J_{T^*} T S^{-1} k_1^{(n)}) + T(T^* k_0^{(n)} + Q_T J_T S^{-1} k_1^{(n)}) = k_0^{(n)},$$

and similarly $(Vl^{(n)})_1 = k_1^{(n)}$. Now because U is a closed extension of V, $Ul = k$.

The most immediate consequence is that $\boldsymbol{R}(U)$ is dense; this was all that was lacking to complete the proof of (b).

But we have showed in addition that U^{-1} is an extension of the operator

$$(3.7) \qquad \left[\begin{array}{ccc|c|cc} \ddots & & & & & \\ & O & 1 & & & \\ & & O & J_{T^*} Q_{T^*} & -J_{T^*} T & \\ \hline & & & T^* & Q_T J_T & \\ \hline & & & & O & 1 \\ & & & & & O \\ & & & & & & \ddots \end{array}\right],$$

which is interpreted similarly to (3. 3). By the same reasoning used in connection with (3. 5), we see that $\bigvee \{U^{-n}\mathfrak{H}\colon n = 0, 1, 2, \ldots\} = \cdots \oplus S^{-2} J_{T^*}\mathfrak{H} \oplus S^{-1} J_{T^*}\mathfrak{H} \oplus \mathfrak{H}$, and this completes the proof of (d). We also see at once that $P_{\mathfrak{H}} U^{-n}|\mathfrak{H}$ is an extension of T^{*n} $(n = 1, 2, \ldots)$.

To complete the proof of (1. 1), it remains to show that $\boldsymbol{D}(U^{-n}) \cap \mathfrak{H} = \boldsymbol{D}(T^{*n})$. It suffices, just as before, to check that when $k_1 = 0$, we have $k \in \boldsymbol{D}(U^{-1})$ if and only if $k_0 \in \boldsymbol{D}(T^*)$. Let, then, $k \in \boldsymbol{R}(U)$, $k_1 = 0$. It is easy to see that we may assume $k \in \mathfrak{H}$ (i.e., $k = k_0$) without loss of generality. Because U is the minimal closed extension of V, we may take $k = Ul$, with sequences $(l^{(n)})$, $(k^{(n)})$ having the properties $k^{(n)} = Ul^{(n)}$, $l^{(n)} \to l$, $k^{(n)} \to k$, $Sl_{-1}^{(n)} \in \boldsymbol{D}(T^*)$, $l_0^{(n)} \in \boldsymbol{D}(T)$. Now $S^{-1} k_1^{(n)} = -J_T T^* S l_{-1}^{(n)} + Q_T l_0^{(n)}$ approaches $S^{-1} k_1 = 0$. From this we want to prove that

$$k_0 = \lim k_0^{(n)} = \lim (Q_{T^*} S l_{-1}^{(n)} + T l_0^{(n)})$$

is in $\boldsymbol{D}(T^*)$. Take any $h \in \boldsymbol{D}(T)$; it will be sufficient to prove that $(k_0, Th) = (l_0, h)$, and for this it will be enough to prove that

$$(3.8) \qquad (k_0^{(n)}, Th) - (l_0^{(n)}, h) \to 0.$$

Substituting the expression for $k_0^{(n)}$, and using (2. 1) and (2. 3),

$$\begin{aligned} (k_0^{(n)}, Th) &= (Q_{T^*} S l_{-1}^{(n)}, Th) + (T l_0^{(n)}, Th) \\ &= (Q_{T^*} S l_{-1}^{(n)}, Th) - (J_T Q_T l_0^{(n)}, Q_T h) + (l_0^{(n)}, h) \\ &= (T^* S l_{-1}^{(n)}, Q_T h) - (J_T Q_T l_0^{(n)}, Q_T h) + (l_0^{(n)}, h). \end{aligned}$$

Hence the left-hand member of (3. 8) is equal to

$$(T^* S l_{-1}^{(n)} - J_T Q_T l_0^{(n)}, Q_T h) = -(J_T S^{-1} k_1^{(n)}, Q_T h) \to -(J_T S^{-1} k_1, Q_T h) = 0,$$

as required. This completes the proof of the dilation property, and thereby that of the Theorem.

Corollary. *Under the hypotheses of the Theorem, there exists a Hilbert space* $\mathfrak{K}_+ \supseteqq \mathfrak{H}$ *and there exists a closed, densely defined operator* U_+ *in* $\mathfrak{K}_+$, *with the following properties:*

(a) $\mathfrak{K}_+$ *is a J-space, with* $\mathfrak{H} \subseteqq J^+(\mathfrak{K}_+)$;
(b) U_+ *is J-isometric, that is,* (1. 2) *holds*;
(c) T^* *is the restriction of* U_+^* *to* $\boldsymbol{D}(T^*)$;
(d) $\bigvee \{U_+^n \mathfrak{H}\colon n=0, 1, 2, \ldots\} = \mathfrak{K}_+$.

Namely, use the construction of the Theorem, and the space $\mathfrak{K}_+$ defined there (3. 5). It inherits the *J*-space structure of $\mathfrak{K}$, because $J|\mathfrak{K}_+$ is still a unitary hermitian operator; the same symbol may be used for this restriction. As U_+ we must of course take $U|\mathfrak{K}_+$. All the assertions of the Corollary follow at once from what has already been proved, except (c). But (c) follows from the definition of adjoint. Indeed, let $h \in \boldsymbol{D}(T^*)$ and $k \in \boldsymbol{D}(U_+)$ be given. As observed in the proof of (c) of the Theorem, $k_0 \in \boldsymbol{D}(T)$. Therefore

$$(U_+ k, h) = (Tk_0, h) + (SQ_T k_0, h) + \big(S(k-k_0), h\big) = (Tk_0, h) = (k_0, T^* h),$$

which is what is needed to prove that $T^* \subseteqq U_+^*$. The Corollary is established.

The Corollary does say, as promised in the Introduction, that every operator has an extension which is the adjoint (or the *J*-adjoint) of a *J*-isometry, but the operator about which it says so is T^*. Indeed, the extension in question is U_+^*; its ordinary adjoint U_+ is a *J*-isometry by (b); but then so is its *J*-adjoint JU_+J, because for all $k, k' \in \boldsymbol{D}(JU_+J)$ we have $Jk, Jk' \in \boldsymbol{D}(U_+)$ and

$$[JU_+Jk, JU_+Jk'] = [U_+Jk, U_+Jk'] = [Jk, Jk'] = [k, k'].$$

4. Geometry of the dilation space

The construction of § 3 carries over more than the algebraic manipulations from the contraction case. I will now exhibit the generalization of the geometric considerations related to the defect spaces [18, I. 3—4]. The geometry here must of course be richer, but it stays in close analogy.

Consider the following subspaces of $\mathfrak{H}$:

$$\mathfrak{D}_T = \overline{\boldsymbol{R}(D_T)} = \overline{\boldsymbol{R}\big((1-T^*T)^+\big)}, \text{ the “defect space” for } T;$$

(4. 1) $$\mathfrak{O}_T = \boldsymbol{N}(1-T^*T), \text{ the “isometric-like space” for } T;$$

$$\mathfrak{X}_T = \overline{\boldsymbol{R}(X_T)} = \overline{\boldsymbol{R}\big((1-T^*T)^-\big)}, \text{ the “excess space” for } T.$$

Defect, isometric-like, and excess spaces for T^* are defined the same way. It is clear that

$$\mathfrak{H} = \mathfrak{D}_T \oplus \mathfrak{O}_T \oplus \mathfrak{X}_T = \mathfrak{D}_{T^*} \oplus \mathfrak{O}_{T^*} \oplus \mathfrak{X}_{T^*}, \tag{4.2}$$

and that the defect and isometric-like spaces have simple characterizations:

$$h \in \mathfrak{O}_T \Leftrightarrow h \in \boldsymbol{D}(T) \text{ and } |T|h = h; \quad h \in \mathfrak{D}_T \Leftrightarrow h \in \bigcap_n \boldsymbol{D}(|T|^n) \text{ and } |T|^n h \to 0. \tag{4.3}$$

It is also clear from definitions that $\mathfrak{D}_T = J_T^+(\mathfrak{H})$, $\mathfrak{X}_T = J_T^-(\mathfrak{H})$, and $\mathfrak{D}_T \oplus \mathfrak{X}_T = J_T(\mathfrak{H})$; and then from (3. 2) and (3. 3), we see that $\mathfrak{K}_+$ (defined by (3. 5)) has the following subspaces invariant for U:

$$\oplus\{S^n\mathfrak{D}_T\colon n = 0, 1, 2, \ldots\} \subseteqq J^+(\mathfrak{K}) \cap \boldsymbol{D}(U), \quad \oplus\{S^n\mathfrak{X}_T\colon n = 1, 2, \ldots\} \subseteqq J^-(\mathfrak{K}).$$

Symmetrically, we construct a positive and a negative subspace

$$\oplus\{S^{-n}\mathfrak{D}_{T^*}\colon n = 0, 1, 2, \ldots\}, \quad \oplus\{S^{-n}\mathfrak{X}_{T^*}\colon n = 1, 2, \ldots\}$$

invariant for U^{-1}.

The complications occur for $n=0, \pm 1$. To describe the action of U there, note first that

$$\begin{aligned} W(\mathfrak{D}_T) &\subseteqq \mathfrak{D}_{T^*}, & W^*(\mathfrak{D}_{T^*}) &\subseteqq \mathfrak{D}_T, \\ W(\mathfrak{O}_T) &\subseteqq \mathfrak{O}_{T^*}, & W^*(\mathfrak{O}_{T^*}) &\subseteqq \mathfrak{O}_T, \\ W(\mathfrak{X}_T) &\subseteqq \mathfrak{X}_{T^*}, & W^*(\mathfrak{X}_{T^*}) &\subseteqq \mathfrak{X}_T. \end{aligned} \tag{4.4}$$

Indeed, the assertions regarding defect and isometric-like spaces are easily proved using the characterizations (4. 3); then the assertions regarding the excess spaces follow using (4. 2).

Now the action of U within $\mathfrak{H}$ — that is, the action of $T = W|T|$ — may be very complicated indeed, but one portion of the complication is brought into view by (4. 2) and (4. 4): We have two orthogonal decompositions of the space, and a pair of partial isometries relating the two.

The action of U "mixes" $J^+(\mathfrak{K})$ with $J^-(\mathfrak{K})$ only at two places: U takes $S^{-1}(\mathfrak{X}_{T^*}) \cap \boldsymbol{D}(U) \subseteqq J^-(\mathfrak{K})$ into $\mathfrak{X}_{T^*} \oplus S(\mathfrak{X}_T)$, although $\mathfrak{X}_{T^*} \subseteqq J^+(\mathfrak{K})$. Secondly, U takes $\mathfrak{X}_T \cap \boldsymbol{D}(U) \subseteqq J^+(\mathfrak{K})$ into $\mathfrak{X}_{T^*} \oplus S(\mathfrak{X}_T)$, although $S(\mathfrak{X}_T) \subseteqq J^-(\mathfrak{K})$.

One especially simple reducing subspace of T has been studied by Apostol [1] and Durszt [2], extending [14, Thm. I. 3. 2]. In terms of the present paper, one may put the central idea as follows. Among subspaces $\mathfrak{U}$ of $\mathfrak{H}$ such that $T|\mathfrak{U}$ is unitary, there is a maximal one $\mathfrak{U}^\infty$, given by

$$\mathfrak{U}^\infty = \bigcap_{n=0}^{\infty} T^n(\mathfrak{O}_T \cap \mathfrak{O}_{T^*}) \cap \bigcap_{n=1}^{\infty} T^{*n}(\mathfrak{O}_T \cap \mathfrak{O}_{T^*}).$$

The part $T|\mathfrak{U}^\infty$ is called the "unitary part" of T; $T|\mathfrak{H} \ominus \mathfrak{U}^\infty$, the "completely non-unitary part".

This is easily proved using the Theorem of § 3, even if T is unbounded; along with the formula

$$\mathfrak{U}^{\infty} = \bigcap_{n=-\infty}^{\infty} U^n(\mathfrak{D}_T \cap \mathfrak{D}_{T^*}).$$

To be sure, the dilation is of interest only as regards the completely non-unitary part. None of the considerations of § 3 would have been affected if I had constructed $\mathfrak{K}^0$ from copies of $\mathfrak{H} \ominus \mathfrak{U}^{\infty}$ rather than copies of $\mathfrak{H}$.

5. Further remarks

1. In case T is a contraction ($\|T\| \leqq 1$), the construction given in § 3 leads to the same dilation as that of [14, I. 5], with J the identity operator and with U unitary. To see this, one need only compare the two step-by-step.

2. Unlike the case of Sz.-Nagy, the construction given here is not determined essentially uniquely by the conditions of the Theorem. Indeed, assuming that T is neither a contraction nor doubly-expansive ($\|Th\| \geqq \|h\|$ and $\|T^* h'\| \geqq \|h'\|$ for all h, h' in the respective domains), I will show how it can always be modified in a non-trivial way.

For we know, once those two cases are excluded, that either $\mathfrak{D}_T$ and $\mathfrak{X}_T$ are both non-zero, or $\mathfrak{D}_{T^*}$ and $\mathfrak{X}_{T^*}$ are both non-zero. This enables us to find operators $Z^{(n)}$ on $\mathfrak{H}$ such that

$$Z^{(0)} = 1; \quad Z^{(n)} J_T = J_T Z^{(n)} \quad (n = 1, 2, \ldots); \quad Z^{(n)} J_{T^*} = J_{T^*} Z^{(n)} \quad (n = -1, -2, \ldots);$$

$$\|Z^{(n)}\| \leqq M, \quad \|(Z^{(n)})^{-1}\| \leqq M \qquad (n = \pm 1, \pm 2, \ldots);$$

for some n,

$$Z^{(n)} \mathfrak{D}_T \neq \mathfrak{D}_T \text{ (if } n > 0), \quad \text{or} \quad Z^{(n)} \mathfrak{D}_{T^*} \neq \mathfrak{D}_{T^*} \text{ (if } n < 0).$$

Then define $Z = \sum_{-\infty}^{\infty} S^n Z^{(n)} S^{-n}$, a continuous, continuously invertible operator on $\mathfrak{K}^0$, and consider it restricted to $\mathfrak{K}$. Evidently all the properties asserted in the Theorem for U hold also for ZU, as does also the desirable property of having $\mathfrak{K}_+ \ominus \mathfrak{H}$ in its domain. Yet the geometry can be quite different. The conditions of the Theorem determine the structure of $\mathfrak{K}$ only with respect to [,], leaving a great deal of freedom as to the inner product (,).

In order to get a uniqueness assertion we must assume more.

Proposition. *Given $\mathfrak{H}$ and T, let $\mathfrak{K}$ and U be constructed as in the proof of the Theorem. Let $\mathfrak{K}'$ be a J-space (with canonical symmetry still denoted by J) and U' a J-unitary in it, which also satisfy conditions* (a)—(d) *of the Theorem. Assume* *further that* (for $n=1,2,\ldots$) $U'^n(\mathfrak{D}_T)\subseteqq J^+(\mathfrak{K}')$, $U'^n(\mathfrak{X}_T\cap D(T))\subseteqq J^-(\mathfrak{K}')$, $U'^{-n}(\mathfrak{D}_{T^*})\subseteqq J^+(\mathfrak{K}')$, $U'^{-n}(\mathfrak{X}_{T^*}\cap D(T^*))\subseteqq J^-(\mathfrak{K}')$. *Then there exists a unitary and J-unitary map Z' of $\mathfrak{K}$ onto $\mathfrak{K}'$ such that $U'=Z'UZ'^{-1}$.*

The proof exploits property (d) in the same way as in the case of contractions. There is no need to go into details.

3. Naturally the construction was motivated in large part by the hope of finding a geometric approach to characteristic functions for arbitrary operators. Just as Sz.-Nagy and Foiaş [14, VI] exhibit a natural geometric genesis of the characteristic function of a contraction, it is hoped to do the same for more general operators. The role of contractive analytic operator-valued function [14, V—VI] might be played by J-contractive ones [3] (further references in [12]). In fact, since the construction described here was found, this program has made some progress, leading to a geometric treatment of the characteristic functions studied earlier by Sahnovič [12] and Kužel' [9]. This will be described in future papers.

References

[1] C. Apostol, Sur la partie normale d'un ensemble d'opérateurs de l'espace de Hilbert, *Acta Math. Acad. Sci. Hung.*, **17** (1966), 1—4.

[2] E. Durszt, On the unitary part of an operator in Hilbert space, *Acta Sci. Math*, **31** (1970), 87—89.

[3] Ю. П. Гинзбург (Ju. P. Ginzburg), О J-нерастягивающих оператор-функциях, *Dokl. Akad. Nauk SSSR*, **117** (1957), 171—173.

[4] P. R. Halmos, Normal dilations and extensions of operators, *Summa Brasil. Math.*, **2** (1950), 125—134.

[5] И. С. Иохвидов (I. S. Iohvidov), Об ограниченности J-изометрических операторов, *Uspehi Mat. Nauk*, **16:4** (1961), 167—170.

[6] И. С. Иохвидов (I. S. Iohvidov), G-изометрические и J-полуунитарные операторы в гильбертовом пространстве, *Uspehi Mat. Nauk*, **20:3** (1965), 175—181.

[7] М. Г. Крейн, (M. G. Krein), *Введение в теорию индефинитных J-пространств и теорию операторов в этих пространствах*, Вторая летн. Мат. школа, I, 15—92 (Kiev 1965).

[8] М. Г. Крейн, Ю. Л. Шмульян (M. G. Krein, Ju. L. Smul'jan), О плюс-операторах в пространстве синдефинитной метрикой, *Mvt. Issled. Akad. Nauk Moldav. SSR*, **1:1** (1966), 131—161.

[9] А. В. Кужель (A. V. Kužel'), Обобщение теоремы Надя—Фояша о факторизации характеристической оператор-функции, *Acta Sci. Math.*, **30** (1969), 225—234.

[10] J. von Neumann, Über adjungierte Funktionaloperatoren, *Ann. of Math.*, **33** (1932), 294—310.

[11] F. Riesz—B. Sz.-Nagy, *Leçons d'analyse fonctionnelle* (Budapest, 1952).

[12] Л. А. Сахнович (L. A. Sahnovič), Неунитарные операторы с абсолютно непрерывным спектром на единичной окружности, *Dokl. Akad. Nauk SSSR*, **181** (1958), 558—551.

[13] B. Sz.-Nagy, Sur les contractions de l'espace de Hilbert, *Acta Sci. Math.* **15** (1953), 87—92.

[14] B. Sz.-Nagy—C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert* (Budapest, 1967).

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