On the unitary part of an operator on Hilbert space

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Let T be a bounded (linear) operator on a Hilbert space H. If a subspace K of H reduces T and $T|K$ is a unitary operator, then $T|K$ is called a *unitary part* of T. A unitary part of T is *maximal* if it is an extension of every other unitary part of T.

A special case of Theorem 4 in [1] states the existence of the maximal unitary part $T|H_0$ for an arbitrary bounded operator T, and characterizes H_0 as the set of vectors φ of H satisfying

$$T^* TA\varphi = A\varphi = TT^* A\varphi \tag{1}$$

for every finite product A of factors equal to T, T^*, and for $A=I$.

In case T is a contraction (i.e. $\|T\| \leqq 1$), H_0 can be characterized as the set of vectors φ in H for which

$$\|T^n \varphi\| = \|\varphi\| = \|T^{*n} \varphi\| \quad (n = 1, 2, \ldots), \tag{2}$$

cf. [2], [3]. For a contraction T, conditions (2) are obviously equivalent to the following ones:

$$T^{*n} T^n \varphi = \varphi = T^n T^{*n} \varphi \quad (n = 1, 2, \ldots). \tag{2'}$$

In this paper we give a new proof of the existence of H_0 for an arbitrary bounded T, and characterize H_0 in a way which is simpler than (1) and very similar to the characterization (2) in the case of contractions. Also, we give a characterization of the orthogonal complement $H \ominus H_0$. Finally, by giving a counterexample we show that the characterization of H_0 by (2) does not hold true in general if T is not a contraction, not even if T is power-bounded.

Consider an arbitrary bounded operator T on the Hilbert space H. We denote by H^0 the set of vectors φ of H for which:

$$\|T^* T^n \varphi\| = \|T^n \varphi\| = \|\varphi\| = \|T^{*n} \varphi\| = \|TT^{*n} \varphi\| \quad (n=1, 2, \ldots) \tag{3}$$

and by H^1 the subspace spanned by the ranges of the operators

$$T^n(I-TT^*) \quad \text{and} \quad T^{*n}(I-T^* T) \quad (n=0, 1, 2, \ldots). \tag{4}$$

Theorem.

(i) *H^0 is a subspace of H reducing T.*

(ii) *$T|H^0$ is the maximal unitary part of T.*

(iii) $H^1 = H \ominus H^0$.

Proof. Let $\varphi \in H^0$. Then

$$\|(I-T^*T)T^n\varphi\|^2 = \|T^n\varphi\|^2 - 2\|T^{n+1}\varphi\|^2 + \|T^*T^{n+1}\varphi\|^2 = 0 \qquad (n = 0, 1, \dots),$$

i.e.:

$$T^*T^{n+1}\varphi = T^n\varphi \qquad (n = 0, 1, 2, \dots).$$

Repeating this computation with T^* in place of T we get:

$$TT^{*n+1}\varphi = T^{*n}\varphi \qquad (n = 0, 1, 2, \dots).$$

Resuming:

(5) $$T^*T^{n+1}\varphi = T^n\varphi, \quad TT^{*n+1}\varphi = T^{*n}\varphi \qquad (n = 0, 1, 2, \dots).$$

So we have: (3) *implies* (5).

On the other hand, if (5) holds for a vector φ, then

$$\|T^*T^{n+1}\varphi\|^2 = \|T^n\varphi\|^2 = (T^{*n}T^n\varphi, \varphi) = (T^*T^n\varphi, T^{n-1}\varphi) =$$
$$= (T^{n-1}\varphi, T^{n-1}\varphi) = \|T^{n-1}\varphi\|^2 = \dots = \|\varphi\|^2,$$

and analogously

$$\|TT^{*n+1}\varphi\| = \|T^{*n}\varphi\| = \|\varphi\| \qquad (n = 0, 1, 2, \dots).$$

So we have: (3) *is equivalent to* (5).

From (5) it is obvious that H^0 is a subspace of H. In the special case $n=0$ we get from (5)

(6) $$T^*T\varphi = \varphi = TT^*\varphi \qquad (\varphi \in H^0).$$

This fact and (3) show that H^0 is invariant both for T and T^*, i.e., H^0 reduces T.

So (i) is proved.

Clearly (6) implies that $T|H^0$ is unitary on H^0. Suppose that $K(\subset H)$ reduces T and $T|K$ is unitary, and let $\varphi \in K$. In this case (3) holds for φ and consequently, $\varphi \in H^0$. Thus $K \subset H^0$, i.e. our statement (ii) is proved.

As regards (iii), (5) shows that the vector $\varphi \in H$ belongs to H^0 if and only if

$$H \perp (I-T^*T)T^n\varphi \quad \text{and} \quad H \perp (I-TT^*)T^{*n}\varphi \qquad (n = 0, 1, \dots),$$

or equivalently:

$$T^{*n}(I-T^*T)H \perp \varphi \quad \text{and} \quad T^n(I-TT^*)H \perp \varphi \qquad (n = 0, 1, \dots).$$

Thus we have: $\varphi \in H^0$ if φ is orthogonal to the ranges of the operators (4), i.e. if $\varphi \perp H^1$. This gives $H^1 = H \ominus H^0$.

So we finished the proof of the theorem.

As regards the counterexample, let $\{\psi_1, \psi_2\}$ be an orthonormal basis in a two-dimensional Hilbert space H and define T by the matrix

$$\begin{pmatrix} -1 & 0 \\ \sqrt{3} & 0 \end{pmatrix}.$$

We prove that there exists a non-zero vector satisfying (2) and such that the corresponding H_0 is $\{0\}$. Indeed, let $\varphi = \frac{1}{2}\psi_1 + \frac{\sqrt{3}}{2}\psi_2$. An easy computation shows that

$$\|\varphi\| = 1, \quad T\varphi = \psi_1 = T^*\varphi, \quad T^n = (-1)^{n-1}T \qquad (n = 1, 2, \ldots),$$

and consequently

$$\|T^n\varphi\| = \|\varphi\| = \|T^{*n}\varphi\| \qquad (n = 1, 2, \ldots),$$

i.e. (2) is fulfilled.

Next observe that T is not a unitary operator. In order to prove that $H_0 = \{0\}$ it suffices therefore to prove that T is not reduced by any non-trivial subspace. If H had a non-trivial subspace reducing T, then this subspace should be one-dimensional, i.e. spanned by an eigenvector of T. An easy computation shows that the two possible linearly independent eigenvectors of T are ψ_2 and $\frac{1}{2}\psi_1 - \frac{\sqrt{3}}{2}\psi_2$, but none of them spans an invariant subspace of T^*.

References

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