# On immersion of locally bounded curvature

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According to results of J. NASH ([11]) and N. H. KUIPER ([10]) $C^1$-immersions are too general to admit a reasonable generalization of the curvature theory of $C^2$-immersions. The idea to extend the curvature theory to a restricted class is therefore justified and in fact this has been done at first by J. HJELMSLEV ([8]) and G. BOULIGAND ([3]) in case of $C^1$-surfaces in 3-dimensional euclidean space. They were mainly interested in the curvature theory of curves on $C^1$-surfaces, i.e. in generalizations of the theorems of Euler and Meusnier. Later on various related results have been obtained by others[1]). In the first part of this paper a class of $C^1$-immersions of $k$-dimensional manifolds into $n$-dimensional euclidean space is introduced, which will be called *immersions of locally bounded curvature*, and it is shown that in their case the second fundamental tensor can be defined in a way which resembles very much the standard one. In the case $n=3$, $k=2$ similar results have been achieved by H. BUSEMANN and W. FELLER ([5]) and A. V. POGORELOV ([14]) for considerably wider classes with more refined methods. In the second part of the paper the case $k = n-1$, i.e. hypersurfaces of locally bounded curvature are considered. It is shown that the theorem on the uniqueness of $C^2$-hypersurfaces with given first and second fundamental forms generalizes to them which gives another point of considering immersions of locally bounded curvature.

## 1. Preliminaries

Some prerequisites of technical nature are provided in this section.

Let $E^n$ be the $n$-dimensional euclidean space, and $V_k^n$ $(k = 1, \ldots, n-1)$ the euclidean vector space formed by its $k$-vectors. Oriented $k$-dimensional subspaces of $V_1^n$ will be identified, as usual, with simple unit $k$-vectors, consequently the set $S_k^n(\subset V_k^n)$ of simple unit $k$-vectors will stand for the set of oriented $k$-dimensional subspaces of $V_1^n$ as well. Let further $B^n$ be the set of complete orthonormal systems

[1]) An account of related results can be found in BUSEMANN [4].

in $V_1^n$, $Q(n, k)$ the group of isometric isomorphisms of $V_k^n$, and $O(n)$ that of orthogonal $n\times n$ matrices with real entries. The facts which follow are well known. $O(n)$ is a simply transitive right transformation group of $B^n$ with the definition: $b\alpha = \left(\sum_{i=1}^{n}\alpha_{ij}a_i\right)_{j=1,\dots,n}$ for $(a_i)_{i=1,\dots,n} = b\in B^n$, $\|\alpha_{ij}\| = \alpha\in O(n)$. Distance on $B^n$ and $O(n)$ is defined by $\sigma(b', b'') = \left[\sum_{i=1}^{n}(a_i' - a_i'')^2\right]^{\frac{1}{2}}$ and $\mu(\alpha', \alpha'') = \left[\sum_{i,j=1}^{n}(\alpha_{ij}' - \alpha_{ij}'')^2\right]^{\frac{1}{2}}$, respectively. If $b\in B^n$ is fixed then $\Phi_b = b\alpha$ defines a distance preserving map $\Phi_{,n}\colon (O(n), \mu)\to(B^n, \sigma)$. With the above definition, $O(n)$ is a distance preserving transformation group of $B^n$, and $\mu$ is left and right invariant. If $(a_i)_{i=1,\dots,n} = b\in B^n$ is fixed and $\sum_{i=1}^{n}x^i a_i = x\in V_1^n$ then with the definition $\alpha x = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}x^j a_i$ the group $O(n)$ is a left transformation group of $V_1^n$ and since each of these transformations is an isometric isomorphism of $V_1^n$ an isomorphism $\Psi_b\colon O(n)\to Q(n, 1)$ is obtained. But there is the standard homomorphism $\Sigma^k\colon Q(n, 1)\to Q(n, k)$, hence for any fixed $b\in B^n$ a homomorphism $\Lambda_b^k = \Sigma^k\circ\Psi_b\colon O(n)\to Q(n, k)$ is defined. If $X_0^k\in S_k^n$ and $H(X_0^k)\subset Q(n, k)$ is the subgroup of elements which leave $X_0^k$ fixed then the inverse image of $H(X_0^k)$ under $\Lambda_b^k$ is a subgroup $H_b(X_0^k)$ of $O(n)$ and there is a one to one correspondence between the left coset space $O(n)/H_b(X_0^k)$ and $S_k^n$, where the left coset corresponding to $X^k\in S_k^n$ is formed by those elements $\alpha$ of $O(n)$ for which $\Lambda_b^k(\alpha)$ sends $X_0^k$ into $X^k$.

A distance preserving map $\theta\colon (O(n), \mu)\to E^{n^2}$ is defined by $\theta(\alpha) = (x^1, \dots, x^{n^2})$, where $x^l = \alpha_{ij}$ for $l = (i-1)n + j$. If $O(n)$ is considered as a Lie group then $\theta$ is a $C^\infty$-embedding, therefore there is a Riemannian metric on $O(n)$ for which $\theta$ is isometric. Let $\varrho_e$ be the distance function of this Riemannian metric ([1], 124); $\varrho_e(\alpha', \alpha'')$ is equal to the infimum of the length of curves joining $\alpha'$, $\alpha''$ if their length is calculated with respect to the distance function $\mu$ ([5]). Therefore $\varrho_e$ and the above Riemannian metric are left and right invariant ([9], 169—172). If $b\in B^n$, $X_0^k\in S_k^n$ are fixed then a distance function $\bar\varrho_e$ is defined on the left coset space $O(n)/H_b(X_0^k)$ by $\bar\varrho_e(\alpha H_b(X_0^k), \beta H_b(X_0^k)) = \inf\{\varrho_e(\xi, \eta) | \xi\in\alpha H_b(X_0^k), \eta\in\beta H_b(X_0^k)\}$. The one to one correspondence $O(n)/H_b(X_0^k)\leftrightarrow S_k^n$ yields a distance function $d^k$ on $S_k^n$ for which this correspondence will be distance preserving. The distance function $d^k$ does not depend on the particular choice of $b$ and $X_0^k$ and it will be called the *auxiliary metrization of* $S_k^n$.

Lemma 1. 1. *Let G be a compact Lie group, H a subgroup and* $\varrho$ *the distance function of a Riemannian metric on G which is left and right invariant. Let the distance function* $\bar\varrho$ *on the left coset space G/H be defined by*

$$\bar\varrho(\alpha H, \beta H) = \inf\{\varrho(\xi, \eta) | \xi\in\alpha H, \eta\in\beta H\}.$$

*Then there exist* $\delta > 0$ *and* $A > 0$ *such that in case* $\bar{\varrho}(H, \alpha H) < \delta$ *a unique* $\xi \in \alpha H$ *with* $\varrho(\varepsilon, \xi) = \bar{\varrho}(H, \alpha H)$ *exists,* $\varepsilon$ *being the identity in* $G$*; further in case* $\bar{\varrho}(H, \alpha_l H) < \delta$ $(l = 1, 2)$ *the inequality* $\varrho(\xi_1, \xi_2) \leqq A\bar{\varrho}(\alpha_1 H, \alpha_2 H)$ *holds for the corresponding* $\xi_l \in \alpha_l H$. *There is a Riemannian metric on* $G/H$ *with distance function* $\tilde{\varrho} \geqq \bar{\varrho}$.

Proof. The existence of a $\xi \in \alpha H$ with $\varrho(\varepsilon, \xi) = \bar{\varrho}(H, \alpha H)$ for any left coset $\alpha H$ is obvious. Since the given Riemannian metric is left and right invariant, $H$ and its left cosets are totally geodesic submanifolds ([1], 136—137). There is such a $\delta' > 0$ that the spherical neighborhood $U(\vartheta)$ of of $\varepsilon$ with radius $\vartheta$ is convex if $0 < \vartheta \leqq \delta'$ ([1], 246—150). Assume that for some $\alpha H$ with $\bar{\varrho}(H, \alpha H) \leqq \delta'$ there are $\xi', \xi'' \in \alpha H$ with $\xi' \neq \xi''$, $\varrho(\varepsilon, \xi') = \varrho(\varepsilon, \xi'') = \bar{\varrho}(H, \alpha H) = \vartheta$. Since $\xi', \xi'' \in U(\vartheta)$, there is a unique minimizing geodesic arc joining them, which with the exception of its endpoints lies in the interior of $U(\vartheta)$. But $\alpha H$ is totally geodesic, therefore this geodesic arc is in $\alpha H$. By the invariance of the Riemannian metric $\bar{\varrho}(H, \alpha H) = \inf\{\varrho(\varepsilon, \zeta) | \zeta \in \alpha H\}$, therefore $\bar{\varrho}(H, \alpha H) < \vartheta$, which is a contradiction. There is a canonical coordinate system of the first kind $\varphi \colon W \to E^m$ and one of the second kind $\psi \colon W \to E^m$, both defined on the neighborhood $W$ of $\varepsilon$, such that 1) if $g_{ij}$ $(i, j = 1, \dots, m)$ are the components of the fundamental tensor of the given Riemannian metric in the coordinate system $\varphi$, then $g_{ij}(\varphi(\varepsilon)) = \delta_{ij}$; 2) if $\psi(\zeta) = (z^1, \dots, z^m)$ for $\zeta \in W$, then $\psi(\zeta) = (0, \dots, 0, z^{s+1}, \dots, z^m)$ for $\zeta \in H \cap W$; moreover, if $\alpha H \cap W$ is not empty then there is a unique $\bar{\alpha} \in \alpha H \cap W$ such that $\zeta = \bar{\alpha} \cdot \bar{\xi}$ with $\bar{\xi} \in H \cap W$ holds for any $\zeta \in \alpha H \cap W$ and

$$\psi(\bar{\alpha}) = (\bar{a}^1, \dots, \bar{a}^s, 0, \dots, 0), \quad \psi(\bar{\xi}) = (0, \dots, 0, \bar{z}^{s+1}, \dots, \bar{z}^m),$$

$$\psi(\zeta) = (\bar{a}^1, \dots, \bar{a}^s, \bar{z}^{s+1}, \dots, \bar{z}^m);$$

3) if $\varphi(\zeta) = (y^1, \dots, y^m)$ for $\zeta \in W$ and $y^i = \chi^i(z^1, \dots, z^m)$ $(i = 1, \dots, m)$ are the transition functions from $\psi$ to $\varphi$, then $\chi^i(\underset{1}{\underbrace{0}}, \dots, \underset{j-1}{\underbrace{0}}, \underset{j}{\underbrace{t}}, \underset{j+1}{\underbrace{0}}, \dots, \underset{m}{\underbrace{0}}) = \delta_{ij} t$ for $j = s+1, \dots, m$, if $(\underset{1}{\underbrace{0}}, \dots, \underset{j-1}{\underbrace{0}}, \underset{j}{\underbrace{t}}, \underset{j+1}{\underbrace{0}}, \dots, \underset{m}{\underbrace{0}}) \in \psi(W)$ ([9], II. 62—86). Let $\gamma \colon [0, 1] \to G$ be the unique minimizing geodesic arc joining $\varepsilon$ and $\zeta \in W$; then $\gamma$ is given by $\gamma^i(t) = y^i \cdot t$ $(0 \leqq t \leqq 1)$ in the coordinate system $\varphi$. Hence

$$\varrho(\varepsilon, \zeta) = \int_0^1 \left[\sum_{i,j=1}^m g_{ij}(\varphi \circ \gamma(t)) y^i y^j\right]^{\frac{1}{2}} dt = \int_0^1 \left[\sum_{i,j=1}^m g_{ij}(\varphi(\varepsilon)) y^i y^j\right]^{\frac{1}{2}} dt = \left[\sum_{i=1}^m (y^i)^2\right]^{\frac{1}{2}}.$$

Therefore if $\alpha H$ is such a coset that the corresponding $\xi \in \alpha H \cap W$ and $\psi(\xi) = (\bar{a}^1, \dots, \bar{a}^s, \bar{x}^{s+1}, \dots, \bar{x}^m)$, then

$$F_j(\bar{a}^1, \dots, \bar{a}^s, \bar{x}^{s+1}, \dots, \bar{x}^m) = \sum_{i=1}^m \chi^i(\bar{a}^1, \dots, \bar{a}^s, \bar{x}^{s+1}, \dots, \bar{x}^m) \cdot \frac{\partial \chi^i}{\partial z^j}\bigg|_{\psi(\xi)} = 0$$

for $j = s+1, \dots, m$. But

$$\frac{\partial F_j}{\partial z^l} = \sum_{i=1}^{m} \left( \frac{\partial \chi^i}{\partial z^l} \frac{\partial \chi^i}{\partial z^j} + \chi^i \frac{\partial^2 \chi^i}{\partial z^l \partial z^j} \right) \quad \text{and hence} \quad \left. \frac{\partial F_j}{\partial z^l} \right|_{\psi(\varepsilon)} = \delta_{jl}$$

for $j, l = s+1, \dots, m$. Consequently there is a neighborhood $U \subset W$ of $\varepsilon$ with $\left. \frac{\partial (F_{s+1}, \dots, F_m)}{\partial (z^{s+1}, \dots, z^m)} \right|_{\psi(\zeta)} \neq 0$ for $\zeta \in U$. Therefore by the implicit function theorem there are analytic functions $\omega^{s+1}(\bar{a}^1, \dots, \bar{a}^s), \dots, \omega^m(\bar{a}^1, \dots, \bar{a}^s)$ defined on a neighborhood $V$ of the origin in $E^s$ such that

$$F_j(\bar{a}^1, \dots, \bar{a}^s, \omega^{s+1}(\bar{a}^1, \dots, \bar{a}^s), \dots, \omega^m(\bar{a}^1, \dots, \bar{a}^s)) = 0 \qquad (j = s+1, \dots, m)$$

for $(\bar{a}^1, \dots, \bar{a}^s) \in V$ and there are no other solutions of $F_j(\bar{a}^1, \dots, \bar{a}^m, z^{s+1}, \dots, z^m) = 0$ $(j = s+1, \dots, m)$ in $V$. Let $\alpha H$ be such a coset that $\xi \in \alpha H \cap W$ and $(\bar{a}^1, \dots, \bar{a}^s) \in V$. Then $x^j = \omega^j(\bar{a}^1, \dots, \bar{a}^s)$ $(j = s+1, \dots, m)$. Let $\delta'' > 0$ be such that $\xi \in \alpha H \cap W$, $(\bar{a}^1, \dots, \bar{a}^s) \in V$ if $\bar{\varrho}(H, \alpha H) \leqq \delta''$. Put $\delta = \min(\frac{1}{4}\delta', \delta'')$ and assume that $\bar{\varrho}(H, \alpha_1 H) = \vartheta_1 \leqq \bar{\varrho}(H, \alpha_2 H) = \vartheta_2 \leqq \delta$. Since $\varrho(\xi_1, \xi_2) \leqq \frac{1}{2}\delta'$, there is a unique $\xi_2' \in \alpha_2 H$ with $\bar{\varrho}(\alpha_1 H, \alpha_2 H) = \varrho(\xi_1, \xi_2')$. Further there are such bounds $A' \leqq A''$ that

$$A' \left[ \sum_{i=1}^{m} (z_1^i - z_2^i)^2 \right]^{\frac{1}{2}} \leqq \varrho(\zeta_1, \zeta_2) \leqq A'' \left[ \sum_{i=1}^{m} (z_1^i - z_2^i)^2 \right]^{\frac{1}{2}}$$

for $\zeta_1, \zeta_2 \in V$, $\psi(\zeta_l) = (z_l^1, \dots, z_l^m)$ $(l = 1, 2)$ ([5]). Therefore

$$\frac{\varrho(\xi_1, \xi_2)}{\bar{\varrho}(\alpha_1 H, \alpha_2 H)} = \frac{\varrho(\xi_1, \xi_2)}{\varrho(\xi_1, \xi_2')} \leqq \frac{A'' \left[ \sum_{i=1}^{s} (\bar{a}_1^i - \bar{a}_2^i)^2 + \sum_{j=s+1}^{m} (x_1^j - x_2^j)^2 \right]^{\frac{1}{2}}}{A' \left[ \sum_{i=1}^{s} (\bar{a}_1^i - \bar{a}_2^i)^2 + \sum_{j=s+1}^{m} (x_1^j - x_2'^j)^2 \right]^{\frac{1}{2}}} \leqq$$

$$\leqq \frac{A''}{A'} \left[ 1 + \frac{\sum_{j=s+1}^{m} (x_1^j - x_2^j)^2}{\sum_{i=1}^{s} (\bar{a}_1^i - \bar{a}_2^i)^2} \right]^{\frac{1}{2}} \leqq$$

$$\leqq \frac{A''}{A'} \left( 1 + \left[ \frac{\sum_{j=s+1}^{m} (\omega^j(\bar{a}_1^1, \dots, \bar{a}_1^s) - \omega^j(\bar{a}_2^1, \dots, \bar{a}_2^s))^2}{\sum_{i=1}^{s} (\bar{a}_1^i - \bar{a}_2^i)^2} \right]^{\frac{1}{2}} \right).$$

But

$$\frac{|\omega^j(\bar{a}_1^1, \dots, \bar{a}_1^s) - \omega^j(\bar{a}_2^1, \dots, \bar{a}_2^s)|}{\left[ \sum_{i=1}^{s} (\bar{a}_1^i - \bar{a}_2^i)^2 \right]^{\frac{1}{2}}} \leqq \sum_{i=1}^{s} \left\{ \left| \frac{\partial \omega^j}{\partial \bar{a}^i} (\bar{a}_1^1, \dots, \bar{a}_1^s) \right| + \right.$$

$$\left. + \int_0^1 \left| \frac{\partial \omega^j}{\partial \bar{a}^i} (\bar{a}_1^1 + t(\bar{a}_2^1 - \bar{a}_1^1), \dots, \bar{a}_1^s + t(\bar{a}_2^s - \bar{a}_1^s)) - \frac{\partial \omega^j}{\partial \bar{a}^i} (\bar{a}_1^1, \dots, \bar{a}_1^s) \right| dt \right\}.$$

Hence the quantity under square root is bounded and the existence of a bound $A$ with $\varrho(\xi_1, \xi_2) \leqq A\bar{\varrho}(\alpha_1 H, \alpha_2 H)$ for $\bar{\varrho}(H, \alpha_l H) \leqq \delta$ $(l=1, 2)$ follows. A Riemannian metric with distance function $\tilde{\varrho} \geqq \bar{\varrho}$ can be evidently given by the standard construction of a homogeneous Riemannian metric on $G/H$ based on the Haar measure of $G$([1], 136).

Lemma 1. 2. *There is such a bound $B$ that $d^k(X_1^k, X_2^k) \leqq B \cdot \|X_1^k - X_2^k\|$ for any $X_1^k, X_2^k \in S_k^n$, where the norm is taken in the euclidean vector space $V_k^n$.*

Proof. Since $S_k^n$ is a $C^\infty$-submanifold of $V_k^n$ this embedding defines a Riemannian metric on $S_k^n$ which has a distance function $\varrho'$, and admits such a bound $B'$ that $\varrho'(X_1^k, X_2^k) \leqq B' \cdot \|X_1^k - X_2^k\|$ for $X_1^k, X_2^k \in S_k^n$ ([5]). Let $\tilde{\varrho}$ be the distance function provided by Lemma 1. 1, then there is such a $B''$ that $\tilde{\varrho} \leqq B'' \cdot \varrho'$. Consequently $B = B' \cdot B''$ is the bound required.

## 2. Immersions of locally bounded curvature

Immersions of locally bounded curvature are introduced in this section and the basic concepts of the curvature theory of $C^2$-immersions are generalized for them.

Let $f$: $M^k \to E^n$ be a $C^1$-immersion of the $C^1$-manifold $M^k$ and for $p \in M^k$ let $U$ be an oriented neighborhood of $p$ in $M^k$. Then the tangent space $T_q M^k$ for $q \in U$ is mapped by the induced map of the tangent bundles $f_*$: $TM^k \to TE^n$ onto an oriented $k$-dimensional subspace of $T_{f(q)}E^n$ which in turn is mapped by $\exp_{f(q)}$: $T_{f(q)}E^n \to E^n$ onto an oriented $k$-plane $L_q^k$ of $E^n$ which defines a simple unit $k$-vector $X_q^k \in S_k^n$. The immersion $f$ defines a Riemannian metric on $M^k$; let $d$ be its distance function. If there is such a $K_p$ that $\limsup\limits_{q \to p} \dfrac{\|X_q^k - X_p^k\|}{d(p, q)} \leqq K_p$, then $f$ is said to be *of bounded curvature at $p$ with the bound $K_p$*. If $p$ has a neighborhood $V$ such that $f$ is of bounded curvature at every $q \in V$ with the same bound $K_V$, then $f$ is said to be *of locally bounded curvature at $p$ with the bound $K_V$*. If $f$ is of locally bounded curvature at every point of $M$ then it is called an *immersion of locally bounded curvature.*

Lemma 2. 1. *Let the $C^1$-immersion $f$: $M^k \to E^n$ of the $C^1$-manifold $M^k$ be of locally bounded curvature at $p \in M^k$. Then there is a coordinate system $\alpha$: $U \to E^k$ of the $C^1$-manifold $M^k$ defined on a neighborhood $U$ of $p$ such that the second derivatives of the vector valued function $x_\alpha = f \circ \alpha^{-1}$: $\alpha(U) \to E^n$ exist are measurable, and independent of the order derivations almost everywhere on $\alpha(U)$.*

Proof. Let $\pi_p$: $E^n \to L_p^k$ be the orthogonal projection on $L_p^k$. There is a neighborhood $U'$ of $p$ in $M^k$ such that $\alpha = \pi_p \circ f$: $U' \to L_p^k$ yields a coordinate system

of the $C^1$-manifold $M^k$. Let $V$ be the neighborhood of $p$ on which $f$ is of locally bounded curvature with the bound $K_V$ according to the assumption of the lemma. Choose $\delta' > 0$ such that $U(2\delta')$, the spherical neighborhood of $p$ with radius $2\delta'$ taken according to the distance function $d$, is contained in $V$. Then $\dfrac{\|X^k_{q_1} - X^k_{q_2}\|}{d(q_1, q_2)} \leqq K_V$ $d(q_l, p) \leqq \delta'$ $(l = 1, 2)$. In fact, by assuming the contrary and considering successive bisections of a minimizing geodesic arc joining $q_1$, $q_2$, one would arrive at a point of $V$ where $K_V$ cannot be a bound for $f$. Put $\delta = \min\left(\delta', \dfrac{1}{K_V}\right)$, then $U = U(\delta)$, the spherical neighborhood of $p$ with radius $\delta$, is contained in $U'$. To verify the last assertion it suffices to see that there is no $q \in U$ with $\langle X^k_q, X^k_p \rangle = 0$ ([13], 117—119); but this is obvious since $q \in U$ and $\langle X^k_q, X^k_p \rangle = 0$ would imply that $d(p, q) \geqq \dfrac{\sqrt{2}}{K_V}$.
If orthonormal coordinate systems are suitably chosen in $E^n$ and $L^k_p$, then

$$x_\alpha(u^1, \dots, u^k) = \left(u^1, \dots, u^k, x^{k+1}_\alpha(u^1, \dots, u^k), \dots, x^n_\alpha(u^1, \dots, u^k)\right)$$

with $(u^1, \dots, u^k) = \alpha(q)$ for $q \in U'$. Put

$$Y^k(u^1, \dots, u^k) = \frac{\partial x_\alpha}{\partial u^1}\bigg|_{\alpha(q)} \wedge \dots \wedge \frac{\partial x_\alpha}{\partial u^k}\bigg|_{\alpha(q)} \quad \text{and} \quad N(u^1, \dots, u^k) = \|Y^k(u^1, \dots, u^k)\|$$

for $q \in U'$. Let $Q$ be the solid $k$-dimensional cube spanned by the basic vectors of the coordinate system of $L^k_p$; then its inverse image in $L^k_q$ under $\pi_p$ is a solid $k$-dimensional parallelepiped $Q_q$, which is spanned by the vectors $\dfrac{\partial x_\alpha}{\partial u^1}\bigg|_{\alpha(q)}, \dots, \dfrac{\partial x_\alpha}{\partial u^k}\bigg|_{\alpha(q)}$, for $q \in U'$. But $N(u^1, \dots, u^k)$ is equal to the $k$-dimensional volume of $Q_q$ and

$$\langle Y^k(u^1, \dots, u^k), X^k_p \rangle = N(u^1, \dots, u^k) \langle X^k_q, X^k_p \rangle = 1 \quad ([10], 56-57).$$

Therefore

$$|N(u^1_1, \dots, u^k_1) - N(u^1_2, \dots, u^k_2)| = \left| \frac{\langle X^k_{q_2} - X^k_{q_1}, X^k_p \rangle}{\langle X^k_{q_1}, X^k_p \rangle \cdot \langle X^k_{q_2}, X^k_p \rangle} \right| \leqq$$

$$\leqq \frac{\|X^k_{q_2} - X^k_{q_1}\|}{|1 - \frac{1}{2}\|X^k_{q_1} - X^k_p\|^2| \cdot |1 - \frac{1}{2}\|X^k_{q_2} - X^k_p\|^2|} \leqq 4 \cdot \|X^k_{q_2} - X^k_{q_1}\|$$

if $q_1, q_2 \in U$. There is a $C > 0$ such that $d(q_1, q_2)^2 \leqq C^2 \cdot \sum_{i=1}^{k} (u^i_1 - u^i_2)^2$ for $q_1, q_2 \in U'$. Consequently

$$|N(u^1_1, \dots, u^k_1) - N(u^1_2, \dots, u^k_2)| \leqq 4 \cdot K_V \cdot d(q_1, q_2) \leqq 4 \cdot K_V \cdot C \cdot \left[\sum_{i=1}^{k} (u^i_1 - u^i_2)^2\right]^{\frac{1}{2}}$$

for $q_1, q_2 \in U$. Therefore

$$\|Y^k(u_1^1, \ldots, u_1^k) - Y^k(u_2^1, \ldots, u_2^k)\| \leqq N(u_1^1, \ldots, u_1^k) \cdot \|X_{q_1}^k - X_{q_2}^k\| +$$

$$+ |N(u_1^1, \ldots, u_1^k) - N(u_2^1, \ldots, u_2^k)| \leqq 6 \cdot K_V \cdot d(q_1, q_2) \leqq 6 \cdot K_V \cdot C \cdot \left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2\right]^{\frac{1}{2}}$$

if $q_1, q_2 \in U$. This means in other words that $Y^k\colon \alpha(U) \to V_k^n$ is a Lipschitz map. Since

$$\left\| \frac{\partial x_\alpha}{\partial u^i}\bigg|_{\alpha(q_1)} - \frac{\partial x_\alpha}{\partial u^i}\bigg|_{\alpha(q_2)} \right\| \leqq \|Y^k(u_1^1, \ldots, u_k^1) - Y^k(u_2^1, \ldots, u_2^k)\|$$

it follows that $\frac{\partial x_\alpha}{\partial u^i}\colon \alpha(U) \to V_1^n$ is a Lipschitz map as well. Therefore by Rademacher's theorem ([13], 271—272), $\frac{\partial^2 x_\alpha}{\partial u^j \partial u^i}$ $(i, j = 1, \ldots, k)$ exist almost everywhere on $\alpha(U)$ and are measurable. The fact that $\frac{\partial^2 x_\alpha}{\partial u^j \partial u^i} = \frac{\partial^2 x_\alpha}{\partial u^i \partial u^j}$ almost everywhere on $\alpha(U)$ follows by an obvious application of Fubini's theorem.

If the $C^1$-immersion $f\colon M^k \to E^n$ is of locally bounded curvature at $p$ and $\alpha\colon U \to E^k$ is a coordinate system of the $C^1$-manifold $M^k$ on the neighborhood $U$ of $p$ constructed according to the proof of the preceding lemma then $\alpha\colon U \to E^k$ will be called a *distinguished coordinate system.*

Let $f\colon M^k \to E^n$ be a $C^1$-immersion, $NM^k$ its normal bundle, $\pi\colon NM^k \to M^k$ the projection in the normal bundle and $\nu\colon NM^k \to E^n$ the normal map of the immersion; $\pi^{-1}(p) = N_p M^k$ is a euclidean vector space and the restriction of $\nu$ to it is an isometric vector space isomorphism. Assume that $f$ is of locally bounded curvature on the neighborhood $V$ of $p$ with the bound $K_V$. Let $(a_1, \ldots, a_n) = b \in B^n$ be such a base that $a_1 \wedge \ldots \wedge a_k = X_p^k$ and $\Lambda_b^k\colon O(n) \to Q(n, k)$ the corresponding homomorphism. Then $H_b(X_p^k) \subset O(n)$ is the subgroup of elements which leave $X_p^k$ fixed, and by the correspondence $O(n)/H_b(X_p^k) \leftrightarrow S_k^n$ for any $q \in M^k$ there is a left coset $\alpha_q H_b(X_p^k)$ consisting of those elements which send $X_p^k$ to $X_q^k$. Let $\delta > 0$ be the number provided by Lemma 1. 1 for the case $G = O(n)$, $H = H_b(X_p^k)$, $\varrho = \varrho_e$ and $B$ the bound given by Lemma 1. 2. There is a neighborhood $U(\delta')$ of radius $\delta' > 0$ of $p$ such that $U(2\delta') \subset V$ and $\delta' \leqq \frac{\delta}{K_V \cdot B}$. Consequently if $q \in U(\delta')$, then

$$\bar{\varrho}_e\big(H_b(X_p^k), \alpha_q H_b(X_p^k)\big) = d^k(X_p^k, X_q^k) \leqq B \cdot \|X_p^k - X_q^k\| \leqq \delta.$$

Hence by Lemma 1. 1 for any $q \in U(\delta')$ there exists a unique $\xi_q \in \alpha_q H_b(X_p^k)$ with

$\varrho_e(\varepsilon, \xi_q)=\bar{\varrho}_e(H_b(X_p^k), \alpha_q H_b(X_p^k))$. The field of bases $\beta: U \to B^n$ defined on $U=U(\delta')$ by $\beta(q) = b \cdot \xi_q = (\bar{w}_1(q), \dots, \bar{w}_n(q))$ $(q \in U)$ will be called a *distinguished field of bases.* Assume that a distinguished coordinate system $\alpha: U \to E^n$ is given as well. Then $(w_1(u^1, \dots, u^k), \dots, w_n(u^1, \dots, u^k))$, $(u^1, \dots, u^k) \in \alpha(U)$ with $w_i(u^1, \dots, u^k) = w_i(q)$ $(i=1, \dots, n)$ for $(u^1, \dots, u^k) = \alpha(q)$ $(q \in U)$ is called a *coordinate representation of the distinguished field of bases.* If $w \in \pi^{-1}(U)$ then

$$v(w) = \sum_{j=k+1}^{n} t^j \bar{w}_j(q) = \sum_{j=k+1}^{n} t^j w_j(u^1, \dots, u^k) \text{ and } \zeta_{\alpha\beta}(w) = (u^1, \dots, u^k, t^{k+1}, \dots, t^n)$$

defines a coordinate system $\zeta_{\alpha\beta}: \pi^{-1}(U) \to \alpha(U) \times E^{n-k}$ for the normal bundle; this will be called a *distinguished coordinate system of the normal bundle.* The map $z_{\alpha\beta}: \alpha(U) \times E^{n-k} \to E^n$ defined by $z_{\alpha\beta}(u^1, \dots, u^k, t^{k+1}, \dots, t^n) = \sum_{j=k+1}^{n} t^j w_j(u^1, \dots, u^k)$ is called a *distinguished coordinate representation of the normal map.*

Lemma 2. 2. *Let the $C^1$-immersion $f: M^k \to E^n$ be of locally bounded curvature at $p \in M^k$, let $\alpha: U \to E^k$, $\beta: U \to B^n$ be a distinguished coordinate system and a distinguished field of bases on the neighborhood $U$ of $p$. Then the corresponding coordinate representation $z_{\alpha\beta}: \alpha(U) \times B_\vartheta^{n-k} \to E^n$ of the normal map is a Lipschitz map, where $B_\vartheta^{n-k}$ is the solid ball of radius $\vartheta > 0$ at the origin in $E^{n-k}$.*

Proof. Since

$$\frac{\|z_{\alpha\beta}(u_1^1, \dots, u_1^k, t_1^{k+1}, \dots, t_1^n) - z_{\alpha\beta}(u_2^1, \dots, u_2^k, t_2^{k+1}, \dots, t_2^n)\|}{\left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2 + \sum_{j=k+1}^{n} (t_1^j - t_2^j)^2\right]^{\frac{1}{2}}} \leqq$$

$$\leqq \frac{\|x_\alpha(u_1^1, \dots, u_1^k) - x_\alpha(u_2^1, \dots, u_2^k)\|}{\left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2\right]^{\frac{1}{2}}} + \frac{\left[\sum_{j=k+1}^{n} (t_1^j - t_2^j)^2\right]^{\frac{1}{2}}}{\left[\sum_{j=k+1}^{n} (t_1^j - t_2^j)^2\right]^{\frac{1}{2}}} +$$

$$+ \frac{\left\|\sum_{j=k+1}^{n} t_2^j \left(w_j(u_1^1, \dots, u_1^k) - w_j(u_2^1, \dots, u_2^k)\right)\right\|}{\left[\sum_{i=1}^{k} (u_1^i - u_2^i)^2\right]^{\frac{1}{2}}}$$

it sufficies to find bounds for the first and the last term. With the notations of and according to the proof of the preceding lemma

$$\frac{\|x_\alpha(u_1^1,\dots,u_1^k)-x_\alpha(u_2^1,\dots,u_2^k)\|}{\left[\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac12}}=$$

$$=\left\{1+\sum_{j=k+1}^{n}\left(\sum_{i=1}^{n}\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q_1)}\frac{u_2^i-u_1^i}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac12}}+\frac{R_{ji}}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac12}}\right)^2\right\}^{\frac12}\leqq$$

$$\leqq\left\{1+\sum_{j=k+1}^{n}\left(\sum_{i=1}^{k}\left|\frac{\partial x_\alpha^j}{\partial u^i}\right|\bigg|_{\alpha(q_1)}+\frac{|R_{ji}|}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac12}}\right)^2\right\}^{\frac12}\leqq$$

$$\leqq 1+\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left(\left|\frac{\partial x_\alpha^j}{\partial u^i}\right|\bigg|_{\alpha(q_1)}+\frac{|R_{ji}|}{\left[\sum_{s=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac12}}\right)\leqq$$

$$\leqq 1+\sqrt{2}\,\|Y^k(u_1^1,\dots,u_1^k)\|+\frac{1}{\left[\sum_{s=1}^{k}(u_1^s-u_2^s)^2\right]^{\frac12}}\sum_{j=k+1}^{n}\sum_{i=1}^{k}|R_{ji}|.$$

But

$$\|Y^k(u_1^1,\dots,u_1^k)\| = N(u_1^1,\dots,u_1^k)=\frac{1}{\langle X_{q_1}^k, X_p^k\rangle}=\frac{1}{1-\frac12\|X_{q_1}^k-X_p^k\|^2}\leqq 2$$

and

$$R_{ji}=\int_0^1\left(\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q(t))}-\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q_1)}\right)(u_2^i-u_1^i)\,dt,$$

where

$$\alpha(q(t))=(u_1^1+t(u_2^1-u_1^1),\dots,u_1^k+t(u_2^k-u_1^k)),\quad (0\leqq t\leqq 1).$$

Therefore

$$\sum_{j=k+1}^{n}\sum_{i=1}^{k}|R_{ji}|\leqq\int_0^1\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left|\left(\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q(t))}-\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q_1)}\right)(u_2^i-u_1^i)\right|dt\leqq$$

$$\leqq\left[\sum_{s=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac12}\cdot\int_0^1\left[2\sum_{j=k+1}^{n}\sum_{i=1}^{k}\left(\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q(t))}-\frac{\partial x_\alpha^j}{\partial u^i}\bigg|_{\alpha(q_1)}\right)^2\right]^{\frac12}dt\leqq$$

$$\leqq\left[2\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac12}\int_0^1\left\|Y^k(\alpha(q(t)))-Y^k(\alpha(q_1))\right\|dt\leqq$$

$$\leqq\left[2\sum_{i=1}^{k}(u_1^i-u_2^i)^2\right]^{\frac12}\cdot 6\cdot K_V\cdot d(q_1,q_2).$$

Consequently

$$\frac{\|x_\alpha(u_1^1, \dots, u_1^k) - x_\alpha(u_2^1, \dots, u_2^k)\|}{\left[\sum_{i=1}^{k}(u_1^i - u_2^i)^2\right]^{\frac{1}{2}}} \leqq 1 + 8\sqrt{2}.$$

By Lemmas 1.1 and 1.2

$$\left\| \sum_{j=k+1}^{n} t_j\big(w_j(u_1^1, \dots, u_1^k) - w_j(u_2^1, \dots, u_2^k)\big) \right\| \leqq$$

$$\leqq \vartheta \left[ \sum_{j=k+1}^{n} (w_j(u_1^1, \dots, u_1^k) - w_j(u_2^1, \dots, u_2^k))^2 \right]^{\frac{1}{2}} \leqq \vartheta \cdot \sigma(b\xi_{q_1}, b\xi_{q_2}) =$$

$$= \vartheta \cdot \mu(\xi_{q_1}, \xi_{q_2}) \leqq \vartheta \cdot \varrho_e(\xi_{q_1}, \xi_{q_2}) \leqq \vartheta \cdot A \cdot \bar{\varrho}_e\big(\alpha_1 H_b(X_p^k), \alpha_2 H_b(X_p^k)\big) =$$

$$= \vartheta \cdot A \cdot d^k(X_{q_1}^k, X_{q_2}^k) \leqq \vartheta \cdot A \cdot B \cdot \|X_{q_1}^k - X_{q_2}^k\| \leqq \vartheta \cdot A \cdot B \cdot K_V \cdot d(q_1, q_2) \leqq$$

$$\leqq \vartheta \cdot A \cdot B \cdot K_V \cdot C \left[ \sum_{i=1}^{k} (u_1^i - u_2^i)^2 \right]^{\frac{1}{2}}.$$

Theorem 2.1. *Let $f\colon M^k \to E^n$ be an immersion of locally bounded curvature. Then its second fundamental tensor and second fundamental forms exist almost everywhere on $M^k$.*

Proof. Let $\alpha\colon U \to E^k$ be a distinguished coordinate system, $\beta\colon U \to B^n$ a distinguished field of bases. By the preceding lemma $w_i\colon \alpha(U) \to V_1^n$ $(i=1, \dots, n)$ is a Lipschitz map and therefore, according to Rademacher's theorem, $\frac{\partial w_i}{\partial u^s}$ $(s=1, \dots, k)$ exist almost everywhere on $\alpha(U)$ and are measurable. As a consequence the usual definition of the second fundamental tensor applies almost everywhere on $U$ as follows. Let $w(u^1, \dots, u^k) = \sum_{j=k+1}^{n} \beta^j(u^1, \dots, u^k) w_j(u^1, \dots, u^k)$ be the coordinate representation of a normal field, where $\beta^j$ are differentiable functions on $\alpha(U)$, and $v = \sum_{i=1}^{k} \alpha^i \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$, then $D(v, w)$, the derivative of $w$ in the direction of $v$, exists if $\frac{\partial w_j}{\partial u^i}\Big|_{\alpha(q)}$ $(i=1, \dots, k;\ j=k+1, \dots, n)$ all exist. Let $S_q(v, w)$ be the orthogonal projection of $D(v, w)$ on $X_q^k$, then

$$S_q(v, w) = \sum_{i=1}^{k} \sum_{r=1}^{k} \sum_{s=1}^{k} \sum_{j=k+1}^{n} g^{ir}(u^1, \dots, u^k) \alpha^s \beta^j(u^1, \dots, u^k) \left\langle \frac{\partial x_\alpha}{\partial u^r}, \frac{\partial w_j}{\partial u^s} \right\rangle\Big|_{\alpha(q)} \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$$

where $g^{ir}$ are the contravariant components of the first fundamental tensor. Therefore

$S_q(v, w)$ defines a tensor $S_q\colon T_qM^k \times N_qM^k \to T_qM^k$. Consequently a tensor field $S$ is obtained which is defined for almost every $q \in M^k$ and is called the *second fundamental tensor of the immersion.* If $\|w\| = 1$ and $w$ is fixed then $l_w(v, v') = \langle S_q(v, w), v'\rangle = \sum_{r=1}^{k} \sum_{s=1}^{k} \sum_{j=k+1}^{n} \alpha'^r \alpha^s \beta^j \left\langle \frac{\partial x_\alpha}{\partial u^r}, \frac{\partial w_j}{\partial u^s}\right\rangle$ is a bilinear form

$$l_w\colon T_qM^k \times T_qM^k \to E^1$$

in $v = \sum_{i=1}^{k} \alpha^i \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$ and $v' = \sum_{i=1}^{k} \alpha'^i \frac{\partial x_\alpha}{\partial u^i}\Big|_{\alpha(q)}$. This is called the *second fundamental form of the immersion at $q$ in the normal direction* $w \in N_qM^k$.

In case of a hypersurface of locally bounded curvature $f\colon M^{n-1} \to E^n$ there is essentially one choice for $w$, therefore the hypersurface has one second fundamental form $l$. The above theorem and Lemma 2.1 give the following

Corollary. *Let $f\colon M^{n-1} \to E^n$ be a hypersurface of locally bounded curvature and $l$ its second fundamental form, then $l_q\colon T_qM^{n-1} \times T_qM^{n-1} \to E^1$ is symmetric for almost every $q \in M^{n-1}$. The coefficients of $l$ are measurable in any distinguished coordinate system of the hypersurface.*

## 3. The uniqueness of hypersurfaces of locally bounded curvature with given first and second fundamental forms

The uniqueness theorem for $C^\infty$-hypersurfaces is the following: Let the $C^\infty$-hypersurfaces $f_1, f_2\colon M^{n-1} \to E^n$ $(n \geqq 3)$ have common first and second fundamental forms where the second fundamental forms are taken with respect to $C^\infty$-unit normal fields $\bar{w}_n^1, \bar{w}_n^2\colon M^{n-1} \to V_1^n$ of the hypersurfaces $f_1, f_2$. Then there exists a distance preserving transformation $\Phi\colon E^n \to E^n$ with $f_2 = \Phi \circ f_1$ ([7], 79—81).

This theorem will be generalized to hypersurfaces of locally bounded curvature in this section.

Lemma 3.1. *Let the $C^1$-hypersurface $f\colon M^{n-1} \to E^n$ be of locally bounded curvature at $p \in M^{n-1}$ with the bound $K_V$, let $\alpha\colon U \to E^{n-1}$ be a distinguished coordinate system, $\beta\colon U \to B^n$ a distinguished field of bases on the neighborhood $U$ of $p$ and $z_{\alpha\beta}\colon \alpha(U) \times E^1 \to E^n$ the corresponding coordinate representation of the normal map. Then there exist a neighborhood $\tilde{U} \subset U$ of $p$ and numbers $\delta, \vartheta > 0$ such that*

$$\|z_{\alpha\beta}(u_1^1, \dots, u_1^{n-1}, t_1^n) - z_{\alpha\beta}(u_2^1, \dots, u_2^{n-1}, t_2^n)\| \geqq \delta \left[\sum_{i=1}^{n-1} (u_1^i - u_2^i)^2 + (t_1^n - t_2^n)^2\right]^{\frac{1}{2}}$$

*if*

$$(u_l^1, \dots, u_l^{n-1}, t_l^n) \in \alpha(\tilde{U}) \times B_\vartheta^1 \qquad (l = 1, 2).$$

Proof. If $u_1^i = u_2^i$ $(i = 1, \dots, n-1)$, the inequality holds with $\delta = 1$ for any $\vartheta > 0$, therefore it will be assumed that $(u_l^1, \dots, u_l^{n-1})$ $(l = 1, 2)$ correspond to different $q_1, q_2 \in U$. Assume further that $t_1^n \geqq t_2^n$ and put $t = t_2^n$, $\Delta t = t_1^n - t_2^n$,

$$\Delta u = \left[\sum_{i=1}^{n-1} (u_2^i - u_1^i)^2\right]^{\frac{1}{2}}, \quad x = x_\alpha(u_1^1, \dots, u_1^{n-1}) - x_\alpha(u_2^1, \dots, u_2^{n-1}),$$

$$\tilde{w}_l = \overline{w}_n(q_l), \quad y = x + t(\tilde{w}_1 - \tilde{w}_2), \quad z = x + t_1^n \tilde{w}_1 - t_2^n \tilde{w}_2, \quad d = d(q_1, q_2).$$

Let $\varepsilon_l$ be the angle of the vector $x$ and the subspace $X_{q_l}^{n-1}$ and put $\gamma_l = \sphericalangle(x, \tilde{w}_l)$, $\gamma' = \sphericalangle(y, \tilde{w}_1)$, $\gamma'' = \sphericalangle(x, \tilde{w}_1 - \tilde{w}_2)$, $2\gamma = \sphericalangle(\tilde{w}_1, \tilde{w}_2)$ where $\gamma'' = 0$ if $\tilde{w}_1 = \tilde{w}_2$. According to the proof of Lemma 2. 2, $1 \leqq \dfrac{\|x\|}{\Delta u} \leqq 1 + 8\sqrt{2}$ if $q_1, q_2 \in U$. Let $\alpha' : U' \to E^{n-1}$ be a distinguished coordinate system at $q_1 \in U$ and let $q_2 \in U' \cap U$. Then for a coordinate system suitably chosen in $E^n$ we have $|\operatorname{tg} \varepsilon_1| = \left|\dfrac{\Delta x_{\alpha'}^n}{\Delta u'}\right|$ where, like below, primes refer to analogous quentities in case of the coordinate system $\alpha'$. Therefore

$$|\operatorname{tg} \varepsilon_1| = \left|\left(\sum_{i=1}^{n-1} \frac{\partial x_{\alpha'}^n}{\partial u'^i}\bigg|_{\alpha'(q_2)} \frac{u_1'^i - u_2'^i}{\Delta u'} + \frac{R'_{ni}}{\Delta u'}\right)\right| \leqq \sum_{i=1}^{n-1} \left(\left|\frac{\partial x_{\alpha'}^n}{\partial u'^i}\right|\bigg|_{\alpha'(q_2)} + \frac{|R'_{ni}|}{\Delta u'}\right) \leqq$$

$$\leqq \sqrt{2}\, \|Y^{n-1}(u_2'^1, \dots, u_2'^{n-1})\| + \frac{1}{\Delta u'} \sum_{i=1}^{n-1} |R'_{ni}| \leqq$$

$$\leqq \sqrt{2}\, \|Y^{n-1}(u_1'^1, \dots, u_1'^{n-1}) - Y^{n-1}(u_2'^1, \dots, u_2'^{n-1})\| +$$

$$+ \frac{1}{\Delta u'} \sum_{i=1}^{n-1} |R'_{ni}| \leqq \sqrt{2} \cdot 6 \cdot K_V \cdot C' \cdot \Delta u' + \sqrt{2} \cdot 6 \cdot K_V \cdot C' \cdot \Delta u' = 12 \cdot \sqrt{2} \cdot K_V \cdot C' \cdot \Delta u',$$

with respect to the proof of Lemmas 2. 1. and 2. 2. Hence $\dfrac{|\operatorname{tg} \varepsilon_1|}{\|x\|} \leqq \dfrac{|\operatorname{tg} \varepsilon_1|}{\Delta u'} \leqq$ $\leqq 12 \cdot \sqrt{2} \cdot K_V \cdot C'$ if $q_1 \in U$, $q_2 \in U' \cap U$. But the value of $C'$ which may depend on $q_1$ is bounded on a neighborhood of $p$, since $f$ is a $C^1$-hypersurface. Therefore a neighborhood $\tilde{U} \subset U$ of $p$ and a bound $D$ exist with $\dfrac{|\operatorname{tg} \varepsilon_l|}{\|x\|} \leqq D$ for $q_1, q_2 \in \tilde{U}$ $(l = 1, 2)$. Obviously

$$\|z\|^2 = (\Delta t)^2 + \|x\|^2 + t^2 \|\tilde{w}_1 - \tilde{w}_2\|^2 -$$

$$- 2\Delta t \left[\|x\|^2 + t^2 \|\tilde{w}_1 - \tilde{w}_2\|^2 - 2t \|x\| \cdot \|\tilde{w}_1 - \tilde{w}_2\| \cdot \cos \gamma''\right]^{\frac{1}{2}} \cdot \cos \gamma' -$$

$$- 2t \|x\| \cdot \|\tilde{w}_1 - \tilde{w}_2\| \cdot \cos \gamma'.$$

Hence

$$\|z\|^2 \geqq (\Delta t)^2 + \|x\|^2 - t^2 \cdot K_V^2 \cdot d^2 -$$

$$-2\Delta t\left[\|x\|^2 + t^2 \cdot K_V^2 \cdot d^2 + 2t \cdot \|x\| \cdot K_V \cdot d \cdot \cos\gamma'\right]^{\frac{1}{2}} - 2t \cdot \|x\| \cdot K_V \cdot d = (\Delta t)^2 +$$

$$+(\Delta u)^2\left[\frac{\|x\|^2}{(\Delta u)^2} - t^2 \cdot K_V^2 \cdot \frac{d^2}{(\Delta u)^2} - 2\Delta t\,\frac{\|x\|}{\Delta u}\left(\frac{\|x\|}{\Delta u} + t \cdot K_V \cdot \frac{d}{\Delta u}\right)\frac{\cos\gamma'}{\|x\|} - \right.$$

$$\left. - 2t \cdot K_V \cdot \frac{\|x\|}{\Delta u} \cdot \frac{d}{\Delta u}\right].$$

But

$$\frac{\cos\gamma'}{\|x\|} = \frac{\langle \tilde{w}_1, x + t(\tilde{w}_1 - \tilde{w}_2)\rangle}{\|x\| \cdot \|x + t(\tilde{w}_1 - \tilde{w}_2)\|} =$$

$$= \left[\frac{\cos\gamma_1}{\|x\|} + 2t\,\frac{\sin^2\gamma}{\|x\|}\right] \cdot \left[1 + t^2\,\frac{\|\tilde{w}_1 - \tilde{w}_2\|^2}{\|x\|^2} + 2t\left(\frac{\cos\gamma_1}{\|x\|} - \frac{\cos\gamma_2}{\|x\|}\right)\right]^{-\frac{1}{2}}$$

and as obvious calculations show $|\cos\gamma_l| \leqq |\sin\varepsilon_l| \leqq |\mathrm{tg}\,\varepsilon_l|$, $\dfrac{\|\tilde{w}_1 - \tilde{w}_2\|}{\|x\|} \leqq K_V \cdot C$, $\dfrac{\sin\gamma}{\|x\|} = K_V \cdot C$. Therefore $\dfrac{\cos\gamma'}{\|x\|}$ is bounded if $q_1, q_2 \in \tilde{U}$ and $t$ is sufficiently small. The above estimates yield the existence of such $\delta, \vartheta > 0$ that $\|z\| \geqq \delta[(\Delta t)^2 + (\Delta u)^2]^{\frac{1}{2}}$ if $q_1, q_2 \in \tilde{U}$ and $|t_1^n|, |t_2^n| \leqq \vartheta$.

Theorem 3. 1. *Let $f_1, f_2$: $M^{n-1} \to E^n$ be hypersurfaces of locally bounded curvature, $g^1, g^2$ their first fundamental forms, and $l^1, l^2$ their second fundamental forms which are calculated with respect to continuous unit normal fields $\tilde{w}_n^1, \tilde{w}_n^2$: $M^{n-1} \to V_1^n$ of the hypersurfaces. Assume that $g^1 = g^2$ and $l_q^1 = l_q^2$ for almost every $q \in M^{n-1}$. Then there exists a distance preserving transformation $\Phi$: $E^n \to E^n$ such that $f_2 = \Phi \circ f_1$.*

Proof. Let $N^l M^{n-1}$ be the normal bundle and $\gamma_l$: $N^l M^{n-1} \to E^n$ be the normal map of the hypersurface $f_l$ $(l = 1, 2)$. In consequence of the preceding lemma for any $p \in M^{n-1}$ there is a distinguished coordinate system $\alpha_l$: $U \to E^{n-1}$ and a distinguished field of bases $\beta_l$: $U \to B^n$ of $f_l$ $(l = 1, 2)$ on a neighborhood $U$ of $p$ such that if $z_l$: $\alpha_l(U) \times E^1 \to E^n$ is the corresponding representation of the normal map, then $\delta, \vartheta > 0$ and $\tilde{U} \subset U$ exist for which the assertion of the lemma holds. Here $\beta_1, \beta_2$ are chosen so as to give $\overline{w}_n^l(q) = \tilde{w}_n^l(q)$ for $q \in U$. Let $\pi_l$: $N^l M^{n-1}$ be the projection in the bundle and $V^l = \{w \mid w \in \pi_l^{-1}(\tilde{U}),\ \|w\| < \vartheta\}$; then the restriction of $v^l$ to $V^l$ is one-to-one. Consequently there is such a distance function $\varrho^l$ that $v_l$: $(V^l, \varrho^l) \to E^n$ is distance preserving. By convexification of the distance function $\varrho_l$ an intrinsic distance function $\bar{\varrho}^l$ is obtained on $V^l$ ([2], [4], 77). The map $\gamma_l$: $(V^l, \bar{\varrho}^l) \to E^n$ is locally distance preserving. If $w \in V^1$ then there is a unique $w' \in V^2$ such that $\pi_1(w) = \pi_2(w') = q$ and if $\gamma_1(w) = f_1(q) + t^n \overline{w}_n^1(q)$ then $v_2(w') = f_2(q) + t^n \overline{w}_n^2(q)$. Let

$\Psi_{\tilde{U}}: V^1 \to V^2$ be defined by $\Psi_{\tilde{U}}(w)=w'$, then $\Psi_{\tilde{U}}$ is one-to-one and maps $V^1$ onto $V^2$. The following argument will show that $\Psi_{\tilde{U}}: (V^1, \bar{\varrho}^1) \to (V^2, \bar{\varrho}^2)$ is distance preserving. Let $\bar{g}^1_{ab}(u^1, \dots, u^{n-1}, t^n)$ be defined for $a, b=1, \dots, n$, and $(u^1, \dots, u^{n-1}, t^n) \in \alpha_1(\tilde{U}) \times B^1_\vartheta$ by

$$\bar{g}^1_{ab} = \begin{cases} \left\langle \dfrac{\partial z_1}{\partial u^a}, \dfrac{\partial z_1}{\partial u^b} \right\rangle & \text{if} \quad a, b = 1, \dots, n-1 \\ \left\langle \dfrac{\partial z_1}{\partial u^a}, \dfrac{\partial z_1}{\partial t^n} \right\rangle & \text{if} \quad a = 1, \dots, n-1;\ b = n, \\ \left\langle \dfrac{\partial z_1}{\partial t^n}, \dfrac{\partial z_1}{\partial u^b} \right\rangle & \text{if} \quad a = n;\ b = 1, \dots, n-1, \\ \left\langle \dfrac{\partial z_1}{\partial t^n}, \dfrac{\partial z_1}{\partial t^n} \right\rangle & \text{if} \quad a = b = n. \end{cases}$$

Then

$$\bar{g}^1_{ab} = g^1_{ab} + t^n(l^1_{ab} + l^1_{ba}) + (t^n)^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} g^1_{ij} g^{ri}_1 g^{sj}_1 l^1_{ar} l^1_{bs}$$

if $a\, b = 1, \dots, n-1$, where $g^1_{ab}$, $g^{ab}_1$ are the covariant and contravariant components of the first fundamental tensor and the $l^1_{ab}$ coefficients of the second fundamental form of $f_1$ in the coordinate system $\alpha_1$; $\bar{g}^1_{ab}=0$ if $a=1, \dots, n-1$; $b=n$ or $a=n$; $b=1, \dots, n-1$; $\bar{g}^1_{ab}=1$ if $a=b=n$. The functions $\bar{g}^1_{ab}$ are measurable on $\alpha_1(\tilde{U}) \times B^1_\vartheta$. Let $\hat{x}(u^1, \dots, u^{n-1})$ $\hat{w}_n(u^1, \dots, u^{n-1})$ be defined by $\hat{x}(u^1, \dots, u^{n-1}) = f_2(g)$, $\hat{w}_n(u^1, \dots, u^{n-1}) = \overline{w}^2_n(g)$, $(u^1, \dots, u^{n-1}) = \alpha_1(g)$, $(g \in \tilde{U})$ and put $\hat{z}(u^1, \dots, u^{n-1}, t^n) = \hat{x}(u^1, \dots, u^{n-1}) + t^n \hat{w}_n(u^1, \dots, u^{n-1})$ for $(u^1, \dots, u^{n-1}, t^n) \in \alpha_1(\tilde{U}) \times B^1_\vartheta$. Let the $\bar{\hat{g}}^2_{ab}(u^1, \dots, u^{n-1}, t^n)$ be defined for $a, b=1, \dots, n$ and $(u^1, \dots, u^{n-1}, t^n) \in \alpha_1(\tilde{U}) \times B^1_\vartheta$ by

$$\bar{\hat{g}}^2_{ab} = \begin{cases} \left\langle \dfrac{\partial \hat{z}}{\partial u^a}, \dfrac{\partial \hat{z}}{\partial u^b} \right\rangle & \text{if} \quad a, b = 1, \dots, n-1, \\ \left\langle \dfrac{\partial \hat{z}}{\partial u^a}, \dfrac{\partial \hat{z}}{\partial t^n} \right\rangle & \text{if} \quad a = 1, \dots, n-1;\ b = n, \\ \left\langle \dfrac{\partial \hat{z}}{\partial t^n}, \dfrac{\partial \hat{z}}{\partial u^b} \right\rangle & \text{if} \quad a = n;\ b = 1, \dots, n-1, \\ \left\langle \dfrac{\partial \hat{z}}{\partial t^n}, \dfrac{\partial \hat{z}}{\partial t^n} \right\rangle & \text{if} \quad a = b = n. \end{cases}$$

Then

$$\bar{\hat{g}}^2_{ab} = \hat{g}^2_{ab} + t^n(\hat{l}^2_{ab} + \hat{l}^2_{ba}) + (t^n)^2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{r=1}^{n-1} \sum_{s=1}^{n-1} \hat{g}^2_{ij} \hat{g}^{ri}_2 \hat{g}^{sj}_2 \hat{l}^2_{ar} \hat{l}^2_{bs}$$

if $a, b = 1, \ldots, n-1$, where $\hat{g}^2_{ab}, \hat{g}^{ab}_2, \hat{l}^2_{ab}$ are the corresponding quantities of $f_2$ in the coordinate system $\alpha_1$; $\bar{g}^2_{ab}=0$ if $a = 1, \ldots, n-1$; $b=n$ or $a=n$; $b = 1, \ldots, n-1$; $\bar{g}^2_{ab}=1$ if $a=b=n$. The measurability of the functions $\bar{g}^2_{ab}$ follows again by the Corollary to Theorem 2. 1 but a coordinate transformation have to be considered as well. According to assumptions of the theorem $\bar{g}^1_{ab}=\bar{g}^2_{ab}$ $(a, b=1, \ldots, n)$ almost everywhere on $\alpha_1(\tilde{U})\times B^\varrho_1$. If $w_1, w_2 \in V^1$ are sufficiently near then there is an $n$-dimensional parallelepiped $P$ in $v_1(V^1)$ formed by a set $S$ of straight line segments parallel and congruent to the one joining $v_1(w_1), v_1(w_2)$ and such that their endpoints fill two $(n-1)$-dimensional cubes the centres of which are $v_1(w_1)$ and $v_1(w_2)$. Let the arc $\varphi$: $[0, 1] \to V^1$ be such that $v_1 \circ \varphi: [0, 1] \to E^n$ is a linear representation of a segment in $S$, and let the functions $\varphi^h(t)$ $(h=1, \ldots, n;\ 0 \leqq t \leqq 1)$ be defined by $z_1(\varphi^1(t), \ldots, \varphi^n(t)) = v_1 \circ \varphi(t)$ $(0 \leqq t \leqq 1)$. In consequence of Lemma 3. 1 $\varphi^h(t)$ are Lipschitz functions; therefore $\dot{\varphi}(t)$ exist almost everywhere on $[0, 1]$ and are measurable in case of any segment in the set $S$. Obvious applications of Fubini's theorem to the set $P$ yield that there are segments in $S$ arbitrary near to the one joining $v_1(w_1)$, $v_1(w_2)$ such that 1) $\bar{g}^1_{ab}(\varphi^1(t), \ldots, \varphi^n(t))$, $\bar{g}^2_{ab}(\varphi^1(t), \ldots, \varphi^n(t))$ $(a, b=1, \ldots, n)$ are measurable on $[0, 1]$; 2) $\bar{g}^1_{ab}(\varphi^1(t), \ldots, \varphi^n(t)) = \bar{g}^2_{ab}(\varphi^1(t), \ldots, \varphi^n(t))$ $(a, b=1, \ldots, n)$ almost everywhere on $[0, 1]$. The distance of the points $v_1 \circ \varphi(0)$, $v_1 \circ \varphi(1)$ in case of such segments is

$$\int_0^1 \left[\sum_{a,b=1}^n \bar{g}^1_{ab}(\varphi^1(t), \ldots, \varphi^n(t))\dot{\varphi}^a(t)\dot{\varphi}^b(t)\right]^{\frac{1}{2}} dt =$$

$$= \int_0^1 \left[\sum_{a,b=1}^n \bar{g}^2_{ab}(\varphi^1(t), \ldots, \varphi^n(t))\dot{\varphi}^a(t)\dot{\varphi}^b(t)\right]^{\frac{1}{2}} dt.$$

But by Tonelli's theorem the last integral is equal to the length of the curve $\hat{z}(\varphi^1(t), \ldots, \varphi^n(t))$ $(0 \leqq t \leqq 1)$ since it is of bounded variation in consequence of Lemma 2. 2. Furthermore we have $\hat{z}(\varphi^1(0), \ldots, \varphi^n(0)) = v_2 \circ \Psi_{\tilde{U}} \circ \varphi(0)$ and $\hat{z}(\varphi^1(1), \ldots, \varphi^n(1)) = v_2 \circ \Psi_{\tilde{U}} \circ \varphi(1)$, and therefore

$$\varrho^1(\varphi(0), \varphi(1)) \geqq \varrho^2(\Psi_{\tilde{U}} \circ \varphi(0), \Psi_{\tilde{U}} \circ \varphi(1)).$$

Consequently $\varrho^1(w_1, w_2) \geqq \varrho^2(\Psi_{\tilde{U}}(w_1), \Psi_{\tilde{U}}(w_2))$, and changing the role of $f_1, f_2$ in the above argument gives $\varrho^1(w_1, w_2) \leqq \varrho^2(\Psi_{\tilde{U}}(w_1), \Psi_{\tilde{U}}(w_2))$. These imply that $\Psi_{\tilde{U}}$: $(V^1, \varrho^1) \to (V^2, \varrho^2)$ is locally distance preserving and $\Psi_{\tilde{U}}$: $(V^1, \bar{\varrho}^1) \to (V^2, \bar{\varrho}^2)$ is distance preserving. Proceeding in like manner a sequence of neighborhoods $\{\tilde{U}_m\}_{m=1,2,\ldots}$ covering $M^{n-1}$ can be obtained with the corresponding sequences $\{(V^l_m, \bar{\varrho}^l_m)\}_{m=1,2,\ldots}$ of metrized neighborhoods in $N^l M^{n-1}$ and the distance preserving maps $\Psi_m$: $(V^1_m, \bar{\varrho}^1_m)$. The set $V_l = \bigcup_{m=1}^{\infty} V^l_m$ is a neighborhood of the zero section in $N^l M^{n-1}$ and since $\bar{\varrho}^l_{m'}, \bar{\varrho}^l_{m''}$ are equal on $V^l_{m'} \cap V^l_{m''}$ there is an intrinsic distance function $\bar{\varrho}^l$ on $V_l$ which is equal to $\bar{\varrho}^l_m$ on $V^l_m$ $(m=1, 2, ..., ; l=1, 2)$. Further $\Psi_{m'}$ and $\Psi_{m''}$ coincide on $V^1_{m'} \cap V^1_{m''}$, therefore there is a distance preserving map $\Psi\colon (V_1, \bar{\varrho}_1) \to (V_2, \bar{\varrho}_2)$ which coincides with $\Psi_m$ on $V^1_m$ for $m=1, 2, \ldots$. Hence $v_l\colon (V_l, \bar{\varrho}_l) \to E^n$ is locally distance preserving. There is an open subset of $V^1_1$ with compact closure $A$ such that $v_1\colon (A, \bar{\varrho}_1) \to E^n$ and $v_2 \circ \Psi\colon (A, \bar{\varrho}_1) \to E^n$ are distance preserving, therefore there is a distance preserving transformation $\Phi\colon E^n \to E^n$ with $v_2 \circ \Psi = \Phi \circ v_1$ on $A$. Assume that the last equality does not hold on $V_1$. Then there is a $w^* \in V_1$ nearest to $A$ with $v_2 \circ \Psi(w^*) \neq \Phi \circ v_1(w^*)$. But $w^*$ has a neighborhood $V^*$ on which $v_1$, $v_2 \circ \Psi$ are distance preserving, consequently there is a distance preserving transformation $\Phi^*\colon E^n \to E^n$ with $v_2 \circ \Psi = \Phi^* \circ v_1$ on $V^*$. Since $\bar{\varrho}_1$ is intrinsic and locally euclidean there is an open subset $V' \subset V^*$ such that $v_2 \circ \Psi = \Phi \circ v_1$ on $V'$. Consequently $v_2 \circ \Psi = \Phi \circ v_1$ on $V^*$ in contradiction with the above assumption. Therefore $v_2 \circ \Psi = \Phi \circ v_1$ on $V_1$. The restriction of this equality to the zero section in $N^1 M^{n-1}$ yields the assertion of the theorem.

Remark. The assumption of Theorem 3. 1 that there are continuous unit normal fields on the whole manifold $M^{n-1}$ for both hypersurfaces does not mean an essential restriction since if $M^{n-1}$ is not orientable its orientable covering manifold can be considered instead.

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