Translations of regular algebras

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By an *algebra* we shall mean any non-empty set together with a non-empty set of finitary operations. Our further terminology is essentially that of [1].

After MALCEV [4], an algebra is called *regular* if no two congruences of this algebra have a congruence-class in common. We call a mapping φ of the algebra A with set of operations Ω into itself an *elementary translation* of A, if there exists an n-ary operation $\omega \in \Omega$, a natural number i $(1 \leqq i \leqq n)$ and elements $a_j \in A$ $(j = 1, \dots, i-1, i+1, \dots, n)$ such that for any $x \in A$ $x\varphi = a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n \omega$ holds. If we say "derived operation" instead of "operation", then we obtain the definition of a *derived translation.* By *translation* we mean an arbitrary product of finitely many elementary translations. Thus, the identical mapping of A is not necessarily a translation, but it is always a derived translation. The set $T(A)$ of all translations of a given algebra A forms a semigroup of transformations of A, which is a subsemigroup of the semigroup $D(A)$ of all derived translations.

It is known that if any two congruences of an arbitrary algebra in the variety $\mathfrak{A}$ commute, then for any $A \in \mathfrak{A}$ $D(A)$ is a transitive semigroup of transformations. The same also holds in those varieties in which the lattice of congruences of every algebra is distributive [2]. Among other investigations concerning regular algebras, MALCEV has formulated a proposition as follows: in order that an algebra A be regular, it is necessary that $D(A)$ be transitive [4]. (We remark that in MALCEV's text the term "translation" is used for "derived translation".) THURSTON in [3] asserts without proof — with reference to [4] — that if a variety $\mathfrak{A}$ is *regular* (that is, all of its algebras are regular), then $T(A)$ is transitive for all $A \in \mathfrak{A}$.

We are going to show that MALCEV's proposition is not valid in general, and we give some description of regular algebras with intransitive semigroups of derived translations. Then we prove that THURSTON's assertion holds even in a more general form. Finally we characterize regular varieties by a certain property of translations for all algebras in such a variety. We begin with

Theorem 1. *On any set having at least two elements there may be defined a regular algebra with intransitive semigroup of derived translations.*

Proof. Let A be an arbitrary set having at least two elements and let $a \in A$. For every pair of different elements $x, y \in A$ $(x \neq a)$ define on A a unary operation ε_{xy} in the following way: for any $z \in A$ let

(1) $$z\varepsilon_{xy} = \begin{cases} y & \text{if } z = x, \\ a & \text{otherwise.} \end{cases}$$

We shall prove that A, with these operations, will be an algebra with the properties desired. A mapping of A into itself shall be a derived translation if and only if it is identical or it is a product of finitely many operations. By definition a is invariant under any translation of A. Verify that A is regular; for this purpose it is sufficient to prove that A is simple.

Let Φ be a congruence on A having a class C with at least two elements, say c, d. Then $c \not\equiv a$ (Φ) implies the existence of ε_{cd} for which we have $c\varepsilon_{cd} = d$, $d\varepsilon_{cd} = a$, according to (1). Clearly, $c\varepsilon_{cd} \equiv d\varepsilon_{cd}(\Phi)$, and thus $d \equiv a(\Phi)$, whence $c \equiv a(\Phi)$. This contradiction shows that $a \in C$. Set now e.g. $c \neq a$, and let b be a further element in A. Then $c\varepsilon_{cb} = b$ and $a\varepsilon_{cb} = a$. Since $c\varepsilon_{cb} \equiv a\varepsilon_{cb}(\Phi)$, we have $b \equiv a(\Phi)$, that is, $b \in C$. Hence $C = A$, and the simplicity of A is verified.

In the following we write $a \to b$ $[a \Rightarrow b]$ for $a, b \in A$ if there exists a $\tau \in T(A)$ $[\delta \in D(A)]$ such that $a\tau = b$ $[a\delta = b]$. It is clear that $a \to b$ implies $a \Rightarrow b$. The corresponding complementary relations will be denoted by $a \not\to b$ and $a \not\Rightarrow b$, respectively.

Our next theorem gives some information about the structure of any regular algebra A with intransitive $D(A)$. To arrange the proof conveniently we first formulate several lemmas.

Lemma 1. *In any algebra, the relation* $\to$ *is transitive.*

Lemma 2. *In any algebra, the relation* $\Rightarrow$ *is the reflexive closure of* $\to$.

These lemmas are obvious from definitions.

Lemma 3. *For any regular algebra A having at least three elements and for arbitrary $a, b, c \in A$ with $a \neq b$, there exists $\tau \in T(A)$ such that either $a\tau = c$, $b\tau \neq c$, or $a\tau \neq c$, $b\tau = c$.*

Proof. We shall call a triplet (x, y, z) *proper* if x, y, z are pairwise different. In the case when (a, b, c) is proper, this lemma is implicit in Malcev [4].

Let now (a, b, c) be not proper. Assume $b = c$ and let $d \in A$, (a, b, d) proper. Then there exists a $\tau_1 \in T(A)$ for which either

(2) $$a\tau_1 = d, \quad b\tau_1 = d_1 \neq d$$

or

(3) $$a\tau_1 = d_1 \neq d, \quad b\tau_1 = d.$$

If (d, d_1, c) is not proper, then $d_1 = c$ and thus exactly one of $a\tau_1$ and $b\tau_1$ equals c. If (d, d_1, c) is proper there exists a $\tau_2 \in T(A)$ for which either

$$d\tau_2 = c, \quad d_1\tau_2 \neq c, \tag{4}$$

or

$$d\tau_2 \neq c, \quad d_1\tau_2 = c. \tag{5}$$

Take $\tau = \tau_1\tau_2$. Now (2) and (4) as well as (3) and (5) imply $a\tau = c$, $b\tau \neq c$; (2) and (5) as well as (3) and (4) imply $a\tau \neq c$, $b\tau = c$.

Lemma 4. *Let A be a regular algebra having at least three elements $a, b, c \in A$ with $a \neq b$. If $a \not\to c$, then $b \to c$.*

Proof. Obvious from Lemma 3.

Theorem 2. *For any algebra A having at least two elements the following statements are equivalent:*

I. *A is regular and $D(A)$ is intransitive.*

II. *A is regular and $T(A)$ is intransitive.*

III. *A is simple and there exists an $a \in A$ such that, for any different $x, y \in A$, $x \not\to y$ holds if and only if $x = a$, $y \neq a$.*

IV. *A is simple and there exists an $a \in A$ such that, for any different $x, y \in A$, $x \not\Rightarrow y$ holds if and only if $x = a$, $y \neq a$.*

Proof. Clearly, I implies II, and IV implies I. Further, III implies IV by Lemma 2. The only non-trivial part of theorem is that III is a consequence of II.

This is immediate in the case when A has exactly two elements. In the other case, let $a, b \in A$, $a \not\to b$. By Lemma 4, $c \to b$ for any $c \in A$ $(c \neq a)$, whence $a \not\to c$ by Lemma 1. Thus $a \to d$ $(d \in A)$ implies $d = a$. Hence by Lemma 4 for each pair $x, y \in A$ $(x \neq a)$ we have $x \to y$.

Let now Φ be an arbitrary congruence on A which has a class consisting of at least two elements. Denote by C the class of including a. It follows from the regularity of A that C contains at least one element e different from a. Choose an element x from A arbitrarily. Then $e \to x$, that is, there exists a translation τ on A such that $e\tau = x$. $e \equiv a(\Phi)$ implies $e\tau \equiv a\tau(\Phi)$, and, because of $a\tau = a$, $x \equiv a(\Phi)$ is valid, whence $x \in C$. Thus $C = A$, and this fact shows that A is simple.

Theorem 2 shows that the example in the proof of Theorem 1 is a typical one.

Theorem 3. *Let A an arbitrary algebra. If $A \times A$ is regular, then $T(A)$ is transitive.*

Proof. We may assume that A has at least two elements. Let $a \not\to b$ for $a, b \in A$, and let $c \in A$, for which $a \to c$ holds. Consider the following relations θ, Φ on $A \times A$:

for any $(x, y), (u, v) \in A \times A$ let $(x, y) \equiv (u, v)(\theta)$ if and only if $x = u$; furthermore, let $(x, y) \equiv (u, v)\ (\Phi)$ if and only if $x = u$ and $a \to x$. It is easy to see, that θ and Φ are congruences on $A \times A$, and $\theta \neq \Phi$; e.g., $(b, b) \equiv (b, c)(\theta)$, but $(b, b) \not\equiv (b, c)(\Phi)$. Nevertheless, θ and Φ have a congruence-class in common, e.g. such a class is formed by the set of all elements of $A \times A$ with c as first component. Thus we see that $A \times A$ is not regular, qu.e.d.

Corollary. (THURSTON [3]) *Let $\mathfrak{A}$ be a regular variety and $A \in \mathfrak{A}$. Then $T(A)$ is transitive.*

Proof. $A \times A \in \mathfrak{A}$, thus $A \times A$ is regular, and hence we may apply Theorem 3.

We remark that the corollary holds not only for varieties, but for classes containing with any algebra its direct product with itself, too.

Finally we show that the condition of Lemma 3 is able to characterize regular varieties.

Theorem 4. *A variety $\mathfrak{A}$ is regular if and only if*

(i) *for any $A \in \mathfrak{A}$ and for all $a, b, c \in A$ $(a \neq b)$ there exists a $\tau \in T(A)$ such that either $a\tau = c$, $b\tau \neq c$, or $a\tau \neq c$, $b\tau = c$.*

Proof. Suppose that $\mathfrak{A}$ regular. If $A \in \mathfrak{A}$ has at least three elements, then (i) follows immediately from Lemma 3. Let A have exactly two elements. We must show that there exists a $\tau \in T(A)$ the range of which contains more than one element. Indeed, in the contrary case the range of any translation of $A \times A$ consists of one element, in contradiction with Lemma 3.

Now suppose that $\mathfrak{A}$ is not regular. This implies, as THURSTON has proved in [3], the existence of an algebra $A \in \mathfrak{A}$ such that the identical congruence and some non-identical congruence Φ have a common class on A. Let c be the unique element of this class, and let $a \equiv b(\Phi)$ $(a \neq b)$. If $a\tau = c$ for $\tau \in T(A)$, then $b\tau \equiv a\tau(\Phi)$, and hence $b\tau = c$ holds. Similarly, $b\tau = c$ implies $a\tau = c$. Thus, (i) is false on $\mathfrak{A}$.

We can observe that Theorem 4 remains valid if we replace the assumption that $\mathfrak{A}$ is a variety with a weaker requirement, namely that $\mathfrak{A}$ contains with any algebra all its homomorphic images and its direct product with itself.

References

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