# A semiring whose Green's relations do not commute

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We call Green's relations on a semiring $R$ the equivalence relations defined thus: $a\mathscr{L}b$ ($a\mathscr{R}b$) if and only if $a$ and $b$ generate the same principal left (right) ideal, $\mathscr{D}=\mathscr{L}\vee\mathscr{R}$ and $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$. The relations $\mathscr{L}$ and $\mathscr{R}$ commute for a large class of semirings which includes semirings with a commutative addition or with a globally idempotent (or weakly reductive) multiplication. In these cases, they have properties analogous to the properties of Green's relations in semigroup theory and apply to the study of ideals of a semiring (cf. [2]). However $\mathscr{L}$ and $\mathscr{R}$ do not commute in general and the purpose of this paper is to give a counterexample, which we have been unable to obtain by more elementary methods.

Our example is the semiring $R$ generated by the two elements set $\{b, d\}$ and subject to the relations:

$$b = d^2b+2db+b = b+2bd+bd^2.$$

Since these relations can also be written under the form: $b = d(db+b)+db+b = b+bd+(b+bd)d$, it is clear that in $R$ the relation: $db+b\ \mathscr{L}b\mathscr{R}\ b+bd$ holds. We shall prove that there exists no element $e$ of $R$ such that $db+b\ \mathscr{R}e\mathscr{L}\ b+bd$.

**1.** It is possible to construct $R$ as the quotient of the free semiring $F$ on the set $\{b, d\}$ by the smallest congruence on $F$ containing the binary relation $\mathscr{G}$ consisting of the two pairs:

(1) $$(b, d^2b+2db+b), \qquad (b, b+2bd+bd^2).$$

However, to obtain a suitable description of $R$, we need to refine this construction by using the construction of $F$ itself which we first recall briefly (cf. [3]).

Let $S$ be the free multiplicative semigroup on $\{b, d\}$, i.e. the set of all monomials $x_1x_2\cdots x_n$ ($n>0$, $x_i\in\{b, d\}$. Then consider the free additive semigroup $W$ on $S$ which is the set of all sums $w_1+w_2+\cdots+w_n$ where $n>0$ and $w_i\in S$ with addition defined by juxtaposition. The multiplication in $S$ can be extended to a associative multiplication of $W$ in the following way:

$$(w_1+\cdots+w_n)(w_1'+\cdots+w_p') = \sum_{i=1}^{n}\left(\sum_{j=1}^{p} w_iw_j'\right),$$

for all $w_i, w'_j \in S$. We shall denote by $S^1$ ($W^0$) the set resulting from the adjunction of a formal identity to the multiplicative (additive) semigroup $S(W)$.

Finally let $\mathscr{D}$ be the transitive closure of the binary relation $\mathscr{F}$ defined as the set of all pairs having either the form $(w, w)$ with $w \in W$ or $(x, y)$ or $(y, x)$ with

$$x = u + \sum_{i=1}^{m}\left(\sum_{j=1}^{n} w_i w'_j\right) + v, \quad y = u + \sum_{j=1}^{n}\left(\sum_{i=1}^{m} w_i w'_j\right) + v,$$

where $m, n > 0$, $w_i, w'_j \in S$ for all $i, j$ and $u, v \in W^0$. Then $F = W/\mathscr{D}$ is the free semiring on $\{b, d\}$.

We shall now describe $R$ as the quotient of $W$ by a suitable congruence. First let $(w, w') \in \mathscr{G}^*$ if and only if $(w, w') \in \mathscr{G}$ or $(w', w) \in \mathscr{G}$. Observe that the binary relation $\mathscr{G}^*$ consists of four pairs and is symmetric. Then let $\mathscr{B}$ be the binary relation on $W$ defined by:

$$\mathscr{B} = \{(u + swt + v,\ u + sw't + v);\ \ u, v \in W^0,\ \ s, t \in S^1,\ \ (w, w') \in \mathscr{G}^*\}.$$

Also let $P = \mathscr{B} \cup \mathscr{F}$. Then we have the following:

Lemma 1. *The smallest congruence $\mathscr{C}$ on $W$ containing both $\mathscr{G}$ and $\mathscr{D}$ is the transitive closure of $\mathscr{P}$. Furthermore $R = W/\mathscr{C}$.*

Proof. Clearly any congruence on $W$ which contains both $\mathscr{G}$ and $\mathscr{D}$ must also contain $\mathscr{G}^*$, $\mathscr{B}$, $\mathscr{P}$ and therefore the transitive closure $\mathscr{C}'$ of $\mathscr{P}$. Thus $\mathscr{C}' \subseteq \mathscr{C}$. To show the inverse inclusion, we shall successively prove that $\mathscr{C}'$ contains both $\mathscr{G}$ and $\mathscr{D}$, and that $\mathscr{C}'$ is a congruence on $W$.

Trivially $\mathscr{C}'$ contains $\mathscr{G}$; also, since $\mathscr{F} \subseteq \mathscr{P}$, certainly the transitive closure $\mathscr{C}^*$ of $\mathscr{P}$ contains the transitive closure $\mathscr{D}$ of $\mathscr{F}$.

Since both $\mathscr{B}$ and $\mathscr{F}$ are symmetric and $\mathscr{F}$ is reflexive, $\mathscr{C}'$ is an equivalence relation. Also both $\mathscr{B}$ and $\mathscr{F}$ admit the addition of $W$, so does $\mathscr{P}$ whence $\mathscr{C}'$ is an additive congruence on $W$. It is left to show that $\mathscr{C}'$ is also a multiplicative congruence. To this end it suffices to prove that $\mathscr{P}$ has the following property: if $z \in W$ and $(x, y) \in \mathscr{P}$, then $(zx, zy) \in \mathscr{C}'$ and $(xz, yz) \in \mathscr{C}'$.

Observe first that, since $\mathscr{D}$ is a congruence on $W$ containing $\mathscr{F}$, for all $(x, y) \in \mathscr{F}$ and $z \in W$, we have: $(zx, zy) \in \mathscr{D}$ and $(xz, yz) \in \mathscr{D}$. Thus, since $\mathscr{D} \subseteq \mathscr{C}'$, $(zx, zy) \in \mathscr{C}'$ and $(xz, yz) \in \mathscr{C}'$ for all $(x, y) \in \mathscr{F}$, $z \in W$.

On the other hand, let $x = u + swt + v$ and $y = u + sw't + v$ be such that $u$, $v \in W^0$, $s, t \in S^1$ and $(w, w') \in \mathscr{G}^*$. If first $z \in S$, then

$$(zx,\ zu + zswt + zv) \in \mathscr{D}, \quad (zy,\ zu + zsw't + zv) \in \mathscr{D}$$

by distributivity modulo $\mathscr{D}$. Also $(zu + zswt + zv,\ zu + zsw't + zv) \in \mathscr{B}$. Since $\mathscr{C}'$ is an additive congruence containing both $\mathscr{B}$ and $\mathscr{D}$, it follows that $(zx, zy) \in \mathscr{C}'$; similarly $(xz, yz) \in \mathscr{C}'$.

If now $z \in W$ so that $z = z_1 + z_2 + \dots + z_r$ for some $r > 0$ and $z_i \in S$. By the above, $(z_i x, z_i y) \in \mathscr{C}'$ for all $i$. Since

$$zx = z_1 x + z_2 x + \dots + z_r x \quad \text{and} \quad zy = z_1 y + z_2 y + \dots + z_r y,$$

and $\mathscr{C}'$ is an additive congruence, we obtain $(zx, zy) \in \mathscr{C}'$. Similarly $(xz, yz) \in \mathscr{C}'$.

It is routine to check that $R = W/\mathscr{C}$, which completes the proof.

**2.** Let $K$ be the set of all elements $x = x_1 + x_2 + \dots + x_r$ $(r > 0,\ x_i \in S)$ of $W$ having the following two properties:

(A) for all $i$, $x_i = d^{p_i} b d^{q_i}$ for some $p_i, q_i \geqq 0$;

(B) there exists $k$ such that $1 \leqq k \leqq r$, $p_k = q_k = 0$, $p_i > 0$ if $i < k$ and $q_i > 0$ if $i > k$.

It is convenient to write the elements of $K$ under the form $x = x_\lambda + b + x_\varrho$, where $x_\lambda, x_\varrho$ have the obvious meaning. Observe that all monomials of $x_\lambda$ are divisible on the left by $d$ and all monomials of $x_\varrho$ are divisible on the right by $d$.

Lemma 2. *Let $x \in K$ and $x' \in W$ satisfy $(x, x') \in \mathscr{C}$. Then $x' \in K$. Furthermore $(x_\lambda + b, x'_\lambda + b) \in \mathscr{C}$ and $(b + x_\varrho, b + x'_\varrho) \in \mathscr{C}$.*

Proof. Clearly, it is enough to show that $\mathscr{P}$ has these properties. We consider successively the two cases $(x, x') \in \mathscr{F}$ and $(x, x') \in \mathscr{B}$. We may also assume $x \neq x'$.

1) Let $(x, x') \in \mathscr{F}$ and $x \in K$. Then we can write for instance:

$$x = u + \sum_{i=1}^{m} \left( \sum_{j=1}^{n} w_i w'_j \right) + v, \quad x' = u + \sum_{j=1}^{n} \left( \sum_{i=1}^{m} w_i w'_j \right) + v,$$

for some $u, v \in W^0$, $w_i, w'_j \in S$. We shall study only the case when $u, v \in W$, the cases when $u = 0$ or $v = 0$ being simpler. Clearly, since $x$ satisfy (A), and $x, x'$ have same monomials up to the order, then $x'$ satisfy (A) too.

Write $u = u_1 + u_2 + \dots + u_{m'}$, $v = v_1 + v_2 + \dots + v_{n'}$ with $m', n' > 0$, $u_i, v_j \in S$. Since $w_i, w'_j \in W^2$, but $b \notin W^2$, certainly $w_i w'_j \neq b$, so that $x \in K$ implies that either $b = u_k$ for some $k = 1, 2, \dots, m'$ or $b = v_{k'}$ for some $k' = 1, 2, \dots, n'$.

Assume that $b = u_k$ for some $k$; then $x_\lambda = u_1 + \dots + u_{k-1}$ and

$$x_\varrho = u_{k+1} + \dots + u_{m'} + \sum_{i=1}^{m} \left( \sum_{j=1}^{n} w_i w'_j \right) + v.$$

Then set $x'_\lambda = x_\lambda$ and

$$x'_\varrho = u_{k+1} + \dots + u_{m'} + \sum_{j=1}^{n} \left( \sum_{i=1}^{m} w_i w'_j \right) + v.$$

Clearly, since all monomials of $x_\lambda$ are divisible on the left by $d$, so are all monomials of $x'_\lambda$; also $(x_\lambda + b, x'_\lambda + b) \in \mathscr{F}$ trivially. Now, looking at the above expressions of $x_\varrho$ and $x'_\varrho$, we see that $(b + x_\varrho, b + x'_\varrho) \in \mathscr{F}$; in particular $x_\varrho$ and $x'_\varrho$ have the same monomials up to the order which implies that all monomials of $x'_\varrho$ are divisible on the right by $d$, since the monomials of $x_\varrho$ have this property. We therefore obtain that $x' = x_\lambda + b + x'_\varrho$ satisfies $(B)$ whence $x' \in K$.

The case when $b = v_{k'}$ for some $k'$ is treated similarly.

2) Let now $(x, x') \in \mathscr{B}$ and $x \in K$; then $x = u + swt + v$, $x' = u + sw't + v$ for some $u, v \in W^0$, $s, t \in S^1$, $(w, w') \in \mathscr{G}^*$. Two subcases are to be considered:

*a)* If $s \neq 1$ or $t \neq 1$, then $swt \in W^2$ so that no monomial of $swt$ can be equal to $b$. Thus $x \in K$ implies $b = u_k$ for some monomial $u_k$ of $u$ (and hence $u \neq 0$) or that $b = v_{k'}$ for some monomial $v_{k'}$ of $v$ (and hence $v \neq 0$).

If first $u = u_1 + \dots + u_{m'}$ $(m' > 0,\ u_i \in S)$ and $b = u_k$ for some $1 \leqq k \leqq m'$, then set

$$x_\lambda = u_1 + u_2 + \dots + u_{k-1} = x'_\lambda,$$

$$x_\varrho = u_{k+1} + \dots + u_{m'} + swt + v,$$

$$x'_\varrho = u_{k+1} + \dots + u_{m'} + sw't + v.$$

All possible forms of $w \in W$ such that $(w, w') \in \mathscr{G}^*$ for some $w' \in W$ are: $b, b + 2bd + bd^2, d^2b + 2db + b$; hence $sbt$ figures as a monomial of $swt$ in all cases; since $x \in K$, and since $swt$ is a term of a sum equal to $x_\varrho$, $s = d^p$, $t = d^q$ for some $p \geqq 0$ and $q > 0$. Then it is obvious to check on all possible forms of $swt$ and $sw't$ that all their monomials are of the form $d^{p'}bd^{q'}$ with $p' \geqq 0$ and $q' > 0$. Since the set of monomials of $x_\varrho$ and $x'_\varrho$ can differ only by monomials of $swt$ and $sw't$, $x \in K$ implies that $x' \in K$. Obviously $(x_\lambda + b, x'_\lambda + b) \in \mathscr{P}$ since $x_\lambda = x'_\lambda$; also $(b + x_\varrho, b + x'_\varrho) \in \mathscr{B}$ is obvious on the form of $x_\varrho$ and $x'_\varrho$.

The case when $b = v_{k'}$ for some monomial $v_{k'}$ of $v$ is treated in a similar way.

*b)* If finally $s = t = 1$, then the different possible forms of $x \in K$ are: $u + b + v$, $u + b + v$, $u + d^2b + 2db + b + v$, $u + b + 2bd + bd^2 + v$; they correspond respectively to the following forms of $x'$: $u + d^2b + 2db + b + v$, $u + b + 2bd + bd^2$, $u + b + v$, $u + b + v$. It is then easy to check on these forms that the result holds also in this case.

The following lemma will be needed later on:

Lemma 3. *Let $x = x_1 + x_2 + \dots + x_r$, with $r > 0$ and $x_i \in S$ for all $i$, be such that $(db + b, x) \in \mathscr{C}$. Then the two sets $A_x = \{i;\ x_i = bd\}$ and $B_x = \{j;\ x_j = dbd\}$ have an even cardinality.*

Proof. The result holds certainly for $x = db + b$, for then $A_x = B_x = \emptyset$. Clearly, it is enough to show that if $(x, x') \in \mathscr{P}$ and $|A_x|$ and $|B_x|$ are even, so are $|A_{x'}|$ and $|B_{x'}|$.

This is clear if $(x, x') \in \mathscr{F}$, for then $x$ and $x'$ have the same monomials up to the order.

Let now $(x, x') \in \mathscr{B}$ so that $x = u + swt + v$, $x' = u + sw't + v$ for some $u, v \in W^0$,

$s=d^p$, $t=d^q$ with $p, q \geqq 0$ (for $x, x' \in K$) and $(w, w') \in \mathcal{G}^*$. Then we have: $|A_x| = |A_u| + |A_{swt}| + |A_v|$, $|B_x| = |B_u| + |B_{swt}| + |B_v|$; also $|A_{x'}| = |A_u| + |A_{sw't}| + |A_v|$, $|B_{x'}| = |B_u| + |B_{sw't}| + |B_v|$. Thus it is enough to show that $|A_{swt}|$ and $|A_{sw't}|$ ($|B_{swt}|$ and $|B_{sw't}|$) differ only by an even number. This is done by direct inspection of the pairs $(swt, sw't)$. Observe that it is enough to consider pairs $(w, w') \in \mathcal{G}$ (since the result is symmetric in $x$ and $x'$) and integers $p, q \leqq 1$. The cases which are left to study are given by the following table:

| | $swt$ | $sw't$ |
|---|---|---|
| $s = t = 1$ | $b$ | $b+2bd+bd^2$ |
| | $b$ | $d^2b+2db+b$ |
| $s = 1,\ t = d$ | $bd$ | $bd+2bd^2+bd^3$ |
| | $bd$ | $d^2bd+2dbd+bd$ |
| $s = d,\ t = 1$ | $db$ | $db+2dbd+dbd^2$ |
| | $db$ | $d^3b+2d^2b+db$ |
| $s = t = d$ | $dbd$ | $dbd+2dbd^2+dbd^3$ |
| | $dbd$ | $d^3bd+2d^2bd+dbd$ |

It is clear by the table that $|A_{swt}|$ and $|A_{sw't}|$($|B_{swt}|$ and $|B_{sw't}|$) differ only by an even number which completes the proof.

**3.** Let $a = db+b$ and $c = b+bd$. Also denote by $\pi$ the canonical projection of $W$ to $R = W/\mathcal{C}$. We wish to show that there exists no element $y$ of $W$ such that the relation $\pi(a)\ \mathcal{R}\pi(y)\ \mathcal{L}\pi(c)$ holds in $R$. Assume that on the contrary such an element $y$ exists.

Since $\pi(a)$ and $\pi(y)$ generate the same principal right ideal of $R$, there exist $y_1, y_2, \ldots, y_r, a_1, a_2, \ldots, a_{r'} \in W^1$, (where $W^1$ is obtained by adjunction of a formal identity to $W$) such that:

$$\left(y, \sum_{i=1}^{r} ay_i\right) \in \mathcal{C} \quad \text{and} \quad \left(a, \sum_{j=1}^{r'} ya_j\right) \in \mathcal{C}. \tag{2}$$

Using the distributivity modulo $\mathcal{C}$, we may first assume that $y_i, a_j \in S^1$ for all $i, j$; also it follows from (2) that

$$\left(a, \sum_{j=1}^{r'} \left(\sum_{i=1}^{r} ay_i a_j\right)\right) \in \mathcal{C}. \tag{3}$$

Lemma 4. *With the above notation, $y_1 = a_1 = 1$; furthermore, for all $i, j > 1$, $y_i = d^{q_i}$ and $a_j = d^{q'_j}$ for some $q_i, q'_j > 0$. Finally, $\sum_{i=1}^{r} ay_i \in K$.*

Proof. Let $x = \sum_{j=1}^{r'} \left( \sum_{i=1}^{r} (d\,by_i a_j + by_i a_j) \right)$. Since $a = db + b$, by distributivity modulo $\mathscr{C}$ and in view of (3), we see that $(a, x) \in \mathscr{C}$. Thus $x \in K$ by Lemma 2. In particular, $by_{i_0} a_{j_0} = b$ for some $i_0, j_0$. Then we must have $i_0 = j_0 = 1$: otherwise there would exist some $i_1 \leqq i_0$, $j_1 \leqq j_0$ such that $i_1 \neq i_0$ or $j_1 \neq j_0$; then $by_1 a_1$ would be a monomial of $x_\lambda$ which is not divisible on the left by $d$; this is impossible for $x \in K$. Therefore $y_1 = a_1 = 1$.

Furthermore, for all $i, j > 1$, $by_1 a_j = ba_j$ and $by_i a_1 = by_i$ are monomials of $x_\varrho$, whence $a_j = d^{q'_j}$ and $y_i = d^{q_i}$ for some $q_i, q'_j > 0$. The last statement follows.

Lemma 5. *For any $y \in W$ such that $\pi(a) \mathscr{R} \pi(y) \mathscr{L} \pi(c)$, $(y, db + b + bd) \in \mathscr{C}$.*

Proof. With the above notation, Lemma 4 implies that $\left( \sum_{i=1}^{r} ay_i \right)_\lambda = db$. Since $\left( y, \sum_{i=1}^{r} ay_i \right) \in \mathscr{C}$ holds by formula (2) and $\sum_{i=1}^{r} ay_i \in K$ by Lemma 4, applying Lemma 2, we obtain $(y_\lambda + b, db + b) \in \mathscr{C}$. A similar reasoning using $\pi(y) \mathscr{L} \pi(c)$ would imply $(b + y_\varrho, b + bd) \in \mathscr{C}$. Since $\mathscr{C}$ is an additive congruence, it follows that

$$(y, db + b + bd) \in \mathscr{C}$$

which completes the proof.

Finally consider $z = \sum_{j=1}^{r'} (dba_j + ba_j + bda_j)$. First Lemma 5 and formula (2) imply that $(a, z) \in \mathscr{C}$. Also, using Lemma 4, it is easy to check that $A_z = \{j; j > 1, q'_j = 1\} \cup \{1\}$ and $B_z = \{j; j > 1, q'_j = 1\}$. Thus $|A_z|$ and $|B_z|$ are of different parity which contradicts $(a, z) \in \mathscr{C}$ in view of Lemma 3. Therefore there exists no $y \in W$ such that $\pi(a) \mathscr{R} \pi(y) \mathscr{L} \pi(c)$ and we have proved:

Theorem 6. *The Green's relations of the semiring $R$ generated by the set $\{b, d\}$ and subject to the relations $d^2 b + 2db + b = b = b + 2bd + bd^2$ do not commute.*

## References


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