On (m, n)-ideals in regular duo semigroups

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Let S be a semigroup. Following the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [1] we shall say that S is regular if $a \in aSa$ for each element a in S. A semigroup S is called duo semigroup if every one-sided ideal of S is a two-sided ideal. We shall prove that in a regular duo semigroup every bi-ideal is a two-sided ideal. More generally we establish that every (m, n)-ideal of a regular duo semigroup is also a two-sided ideal, if m, n are non-negative integers such that $m+n > 0$. For the definition and fundamental properties of (m, n)-ideals we refer to the author's paper [2].

First we need the following result.

Theorem 1. *A non-empty subset A of a regular semigroup S is a bi-ideal of S if and only if there exists a left ideal L and a right ideal R of S such that $A = RL$.*

Proof. Let A be a bi-ideal of a regular semigroup S. We show that it is the product of the smallest right ideal of S containing A and the smallest left ideal of S containing A:

(1) $$A = (A \cup AS)(A \cup SA).$$

It is easy to see that the inclusion

(2) $$A \subseteq (A \cup AS)(A \cup SA) = A^2 \cup ASA$$

holds because $a = axa \in ASA$ for each element a in A. The converse inclusion

(3) $$(A \cup AS)(A \cup SA) \subseteq A$$

also holds since A is a bi-ideal of S.

Conversely we prove that if S is an arbitrary semigroup, L is a left ideal and R is a right ideal of S then the product RL is a bi-ideal of S. It is evident that

(4) $$(RL)(RL) \subseteq RL,$$

that is the product RL is a subsemigroup of S. On the other hand

(5) $$(RL)S(RL) \subseteq RSL \subseteq RL,$$

i.e. RL is a bi-ideal of S. This completes the proof of Theorem 1.

Theorem 2. *Every bi-ideal of a regular duo semigroup is a two-sided ideal.*

Proof. The statement of Theorem 2 is an easy consequence of Theorem 1 because the product of two-sided ideals is also a two-sided ideal.

Corollary. *Every quasi-ideal of a regular duo semigroup is a two-sided ideal.*

This follows from Theorem 2 because every quasi-ideal is a bi-ideal.

Theorem 3. *Suppose that S is a regular duo semigroup and m, n are non-negative integers so that $m+n>0$. Then every (m, n)-ideal of S is a two-sided ideal of S.*

Proof. The statement of Theorem 3 follows from Theorem 2 and from Theorem 1.5 in the author's paper [2] by mathematical induction.

Let S be a semigroup which is a semilattice of groups. It is known that S is a regular duo semigroup (see [1] or [3]). Applying Theorems 2 and 3 we obtain the following results.

Theorem 4. *Let S be a semigroup which is a semilattice of groups. Then every bi-ideal of S is a two-sided ideal.*

Theorem 5. *Suppose that S is a semigroup which is a semilattice of groups and m, n are non-negative integers such that $m+n>0$. Then every (m, n)-ideal of S is a two-sided ideal of S.*

References

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups.* I—II (Providence, 1961; 1967).
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[3] S. Lajos, A csoportok félhálójaként előállítható félcsoportok ideálelméleti jellemzése, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.*, **19** (1969), 113—115.

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