# Bibliographie

**H. Federer, Geometric Measure Theory** (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band 153), XIV + 676 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1969.

This is the first systematic and detailed exposition of the subject, from the foundations up to the most recent results, including many which were not previously published. The book can serve both as an excellent reference book and as a textbook; the reader is merely assumed to be familiar with the very elements of set theory, topology, linear algebra, and commutative ring theory.

The abundance and variety of the material peresented in the book makes an exhausting description in a short review quite impossible. So we can only comment on the general plan of the book, and mention some samples of the most characteristic results contained.

Chapter I contains a systematic account of the methods of multilinear algebra which are used throughout the book. From among these basic tools of geometric measure theory the author uses exterior and alternating algebras to discuss oriented $m$-dimensional vector subspaces of Euclidean $n$-space, and uses symmetric algebras to treat higher differentials of mappings, for example in WHITNEY's theorem or in the theory of strongly elliptic systems of second order partial differential equations.

The techniques of the general theory of integration are developed in Chapter II, the material of which can be warmly recommended to students in a higher course on real analysis. The author's exposition of general measure theory features equally the set theoretic approach of CARATHÉODORY and the function lattice approach of P. J. DANIELL and F. RIESZ. It includes not only the fundamental facts on Lebesgue integration, but also numerous additional topics like the theory of Suslin sets, the theory of covariant measures over homogeneous spaces of locally compact groups, the main properties of Hausdorff-type measures, etc. Connections between integration and linear operations, including the Radon-Nikodým theorem, are discussed in great detail first for an arbitrary lattice of real valued functions, and later for continuous functions on a locally compact space. It then proceeds to give an account of the theory of covering and derivation, which comprises several modern results about derivates based on generalizations of the covering theorems of VITALI and BESICOVITCH. Finally, we find an exhaustive study of CARATHÉODORY's construction of measure, by which he achieved the significant extension of measure theory to lower dimensional subsets of the space, obtaining for example a reasonable notion of area for a two-dimensional surface in Euclidean 3-space.

Chapter III contains the basic facts concerning integration with respect to $m$-dimensional measures over subsets of Euclidean $n$-space. It centers about tangential and rectifiability properties of sets and the transformation formulae corresponding to Lipschitzian maps. Some generalizations of differentiation are defined here; tangent spaces of arbitrary subsets of $R^n$, differentiable submanifolds, etc. are discussed in detail. Then the author exhaustively elaborates the theory of area and coarea of Lipschitzian maps, with a fine application of the coarea formula in integral geometry, and a generalization of the calculus of area and coarea, etc. The rest of the chapter deals with the characterization of rectifiable sets by their projection properties within the framework of structure theory, and with some effect of high order smoothness of functions on the Hausdorff measures of certain associated sets.

In Chapter IV the author employs distributions in the sense of L. SCHWARTZ, and currents as introduced by G. DE RHAM for use in the theory of harmonic forms. The principal objects of the investigation are the normal, rectifiable and integral currents and the integral flat chains. The author follows the lead of H. WHITNEY, *Geometric integration theory* (Princeton, 1957), in using Lipschitzian maps and the notion of mass, but the two books have very different aims, WHITNEY's book being directed to cohomology with general cochains, whereas this one to homology with general chains. This chapter contains a number of topics about integration and differential forms over oriented-sets including the formulae of Gauss, Green and Stokes. The reader will find several versions of these classical results; research on the problem of finding the most natural and general forms of them has greatly contributed to the development of geometric measure theory. Another fundamental result of this chapter is the deformation theorem, which yields basic isoperimetric estimates and links the theory of integral currents to the classical integral homology theory.

Recently the methods of geometric measure theory have led to a very considerable progress in the study of general elliptic variation problems, including the multidimensional problem of least area. Chapter V is entirely devoted to applications of the theory of integral currents to the calculus of variations. The theory is based on the concept of the integral of a positive parametric integrand $\Phi$ of degree $m$ over an $m$-dimensional rectifiable current $T$. The problem of regularity and smoothness of minimizing currents has not yet been completely solved, but some very significant partial results have been obtained, and these are systematically collected here. The study of singularities of minimizing currents is likely to remain a fruitful field of research in years to come. The treatment of strongly elliptic systems of second order partial differential equations is not intended to be as comprehensive as in C. B. MORREY, *Multiple integrals in the calculus of variations* (New York, 1966), but it contains a complete exposition of the results needed for geometric applications. We note that Fourier transformation is not used in the book.

The enumeration of the contents could hardly give a right impression of the richness of the monograph. The presentation is concise, but always clear and well-readable. The last three chapters are in particular useful for those interested in further research work. It is perhaps not exaggerated to assert that this book is of basic importance for everybody who wants to keep pace with up-to-date developments in analysis.

*Ferenc Móricz* (Szeged)

**Ju. M. Berezanskii, Expansions in eigenfunctions of selfadjoint operators** (Translations of Mathematical Monographs, vol. 17) VI+809 pages, American Mathematical Society, Providence, 1968.

Developments in the application of the general theory of selfadjoint operators to spectral problems for differential and difference equations have been very rapid in recent years. Nevertheless, at the time of preparation of the Russian original of the present book, there were no books in existence in which relevant questions were discussed in any complete way. The book of M. A. NAĬMARK (*Linear differential operators;* Moscow, 1954) discusses the spectral theory of selfadjoint ordinary differential and difference equations. Analogous questions for partial differential equations are dealt with in only one chapter of the book of GEL'FAND and ŠILOV (*Generalized functions.* Part III; Moscow, 1968) and in one chapter of the book by DUNFORD and SCHWARTZ (*Linear operators.* Part II; New York, 1963). The author has tried to fill this gap by undertaking the job of giving a comprehensive account of the subject in the form of a monograph.

The reviewer is aware of the fact that a rough chapter-by-chapter description does not do justice to the extremely rich content of the book. The limited space here, however, does not allow to do more.

The nature of the subject makes it necessary to introduce the relevant notions of modern functional analysis. This background is given in a very self-contained manner as an Introduction, which is a chapter by itself. A great deal of discussions in the book are based on generalized functions of finite order. The theory of these functions is given in the first chapter and is formulated in a conveniently abstract form. The shorter Chapter II introduces and discusses the general concepts of the theory of boundary problems. First boundary value problems for linear partial differential equations are considered, then formal schemes for the application of functional methods to the analysis of these problems are presented. Chapter III is devoted to the study of boundary value problems for elliptic equations. Here the chief attention is turned to questions on when each generalized solution of an elliptic equation is sufficiently smooth both in the interior of the domain and up to and including its boundary. These questions are of central importance when we construct expansions with respect to eigenfunctions. Chapter IV contains a number of examples of problems connected with non-elliptic equations, which should be considered as illustrations of the method developed in Chapter II. The discussion of the theory announced in the title of the book actually commences in Chapter V. The general theory of expansions in eigenfunctions (generalized or ordinary) is given here in detail. In this exposition, the main ideas are represented by two methods. The first (due to the author) relies essentially on the use of Radon—Nikodym-type derivatives and the second on the notion of von Neumann's direct integral. The results obtained in the course of studying the general situation are then interpreted for concrete cases such as arbitrary selfadjoint operators in $L^2(G)$, $G\subset R^n$, and Carleman's operators. Chapter VI contains a thorough analysis of the results of the preceding chapter for operators in $L^2$ connected with elliptic equations. A brief summary of the corresponding results for ordinary differential problems is also given here. Chapter VII is devoted to the study of the spectral theory of selfadjoint difference operators in $l^2$. Finally, in Chapter VIII the theory of selfadjoint (differential and difference) operators acting in a space with scalar product generated by a positive definite kernel is constructed. M. G. Krein's results on integral representations for positive definite functions play here a role of central importance.

Each chapter is illuminated with a great many examples, both classical and recent. The book concludes with very instructive bibliographical and historical comments and a very rich bibliography. In this presentation the author has overcome the difficulties inherent in the material treated. Thus the book, in spite of its intricate subject, is quite readable. One of methods for achieving this consists in the author's repeating important concepts and definitions rather than using cross references. The reviewer is convinced that this excellent monograph is to become classic in this field.

The American Mathematical Society has performed a very valuable service in translating the original Russian edition into English. This English edition makes it possible for this very interesting book to appeal to a wider class of interested mathematicians.

*I. Kovács* (Szeged)

**J. L. Bell** and **A. B. Slomson, Models and ultraproducts: an introduction,** ix+322 pages, North-Holland Publishing Company, Amsterdam—London, 1969.

This monograph on ultraproducts appeals to a wide circle of readers. The practising mathematician may be attracted by the accounts giving insight into the major trends of recent development in the field as far as this is possible without an unfavourable increase of the size of the book; the undergraduate will appreciate the fact that everything is presented in a way available also to him; finally it serves everyone's comfort that the author is successful in avoiding the overpredantry that endangers many of the writers on logics in a way that this does not amount to carelessness or looseness in any disturbing quantity. Indeed, the expert reader may supply the subtle points not considered in the text; and, on the other hand, it would not really be to the point to overload the undergraduate reader with refined distinctions where these do not play a prominent role.

The importance of the subject of the book can hardly be overstressed. It is indeed the concept of ultrafilters that directly or indirectly penetrated into widely different fields of mathematics and which — via non-standard analysis — proved that even in disciplines seemingly remote from logics the methods of the latter may turn out very valuable. Though the present book is concerned only with the model-theoretic applications of ultrafilters, it is really hard to imagine this otherwise. These applications form the bases of all others; this is true even for topology — as is shown by non-standard methods — where the use of ultrafilters to replace the convergent sequences of analysis seems in no way to be connected with ultraproducts.

The book begins with introductory chapters on propositional and predicate calculus, model theory with the proof of the Löwenheim-Skolem theorem, compactness theorem, completeness theorem, etc. Another proof, using the maximal ideal theorem instead of the full strength of the axiom of choice, of the compactness theorem is also included. As is known, the existence of such a proof has practical importance in that it makes possible to establish many classical results in analysis as a consequence of a principle weaker than the axiom of choice by using non-standard analysis.

After these basic questions the authors touch upon more specific problems concerning the cardinality of ultraproducts, connections between semantical and algebraic properties of structures, characterization of elementary equivalence with the aid of iterated ultrapowers (ultralimits), completeness and model completeness, algebraically homogeneous and universal structures and saturated models, an extremely useful tool in up-to-date research in model theory; various applications of ultraproducts, among them one to the construction of non-standard models of arithmetic; after considerations of the effect of extending the language so as to include generalized quantifiers the book is concluded with an account of infinitary languages — here a celebrated result of W. Hanf of inconpactness of cardinals is considered, though not in its strongest form, the importance on which is underlined by the fact that it decided (in the negative direction) a long open problem of seemingly pure set-theoretical nature: whether or not it is possible to introduce a countably additive non-atomic measure on the set of all subsets of the first inaccessible cardinal.

The material encompassed in this gap-filling work is still in a boiling stage. Though this means that the final word cannot yet be proclaimed on the subject, it certainly increases the actuality value of this book.

*A. Máté* (Szeged)

**Theory of Finite Groups, A Symposium,** Edited by **Richard Brauer** and **Chih-Han Sah,** XIII + 263 pages, W. A. Benjamin, Inc., New York—Amsterdam, 1969.

This book contains abstracts of lectures held at a symposium on finite groups at Harvard University in 1968. The principal subject of this symposium was one of the most exciting recent problems of finite group theory: the description of simple groups.

After half a century of stagnation, research in the field of finite simple groups got a new impulse in the fifties, owing above all to the new ideas raised by R. Brauer and C. Chevalley to study centralizers of involutions and the Sylow 2-subgroup as well as to apply the algebraic-geometrical method. These ideas have considerably deepened our general knowledge on simple groups, and also led to the discovery of some new types of simple groups. The greatest part of the book is attached to these three ideas. The authors of the 36 abstracts are well-known specialists of finite group theory, among them R. Brauer, W. Feit, D. Gorenstein, Z. Janko, M. Suzuki, J. Tits. The articles contain not only results but also sketch their proofs in a more or less detailed form, so they give abundant and valuable information for the competent reader on the present state of research.

May we mention some samples from the content. M. SUZUKI describes a (new) simple permutation group of order $2^{13}\cdot 3^7\cdot 5^2\cdot 7\cdot 11\cdot 13$ on $2\cdot 3^4\cdot 11$ letters. G. HIGMAN and J. MCKAY prove the existence of a simple group of order $2^7\cdot 3^5\cdot 5\cdot 17\cdot 19$ studied by Z. JANKO. J. L. ALPERIN and R. BRAUER give a description of the possible orders and the possible centralizers of involutions of a simple group with quasi-dihedral Sylow 2-subgroup. CHIH-HAN SAH attacks Schreier's conjecture on the solvability of automorphism groups of simple groups.

In view of the rapid progress in this domain, we can praise the publisher, who, by applying a high-quality multiplying technique, published this precious book very shortly after the symposium.

*Béla Csákány* (Szeged)

**Daniel Ponasse, Logique mathématique,** 164 pages, Office Central de Librairie, Paris, 1967.

The book, an introductory course for post-graduate students, is extremely successful in saying much in a limited space while avoiding any notable adverse effect of its compactness. It presents some fundamental topics in mathematical logics; in particular, it developes the propositional calculus and first-order predicate calculus first from syntactic and then from semantic aspect along parallel lines; gives an insight into the notions of deducibility, deductive systems, completeness, and consistency; proves several equivalent versions of the completeness theorem; considers Boolean rings and algebras with regard to their applications to logical calculi; studies the completeness problem in first-order predicate calculus, the Löwenheim-Skolem theorem and the finiteness principle (compactness theorem) and concludes with an account of first order calculus with equality.

*A. Máté* (Szeged)

**K. Chandrasekharan, Introduction to analytic number theory** (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 148), VIII+140 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1968.

These introductory lectures are prepared with the intention to raise the interest of students and non-specialists in how the analytical means seemingly remote from the field are in fact closely connected with deep problems of number theory. The large amount of literature on the subject understandably makes it difficult to say anything new about the classical results of the subject; through careful selection and grouping of the material the author carried out the undertaken task eminently. After elementary discussions on factorization and congruences the book deals with approximation of irrationals, quadratic residues, arithmetical functions, Chebyshev's estimate for the distribution of primes, uniform distribution modulo 1, Minkowski's theorem on lattice points in convex sets, Dirichlet's theorem on primes in arithmetical progressions and the prime number theorem. The work is complemented with a short historical survey.

*A. Máté* (Szeged)

**C. G. Lekkerkerker, Geometry of numbers** (Series Bibliotheca Mathematica, Vol. 8), 510 pages, Wolters-Noordhoff Publishing, Groningen, North-Holland Publishing Company, Amsterdam—London, 1969.

The geometry of numbers, a subject the basic question of which is when a body in the $n$-dimensional Euclidean space contains an integral point different from the origin, was initiated by a relatively simple theorem of H. MINKOWSKI and its surpising arithmetical applications. The subject has in the last decades aquired a considerable level of development. This monograph intends to satisfy the long-felt need for a comprehensive up-to-date account of the subject. It is natural that such a deep study as this is mainly addressed to practising mathematicians or more advanced students; nevertheless, since only a minimal amount of preliminary knowledge is required; everyone interested in the subject can make use of this book.

The material is so arranged that orientation on the present state of knowledge is very quick; this is achieved by laying stress on problems and not on the methods used in the solutions of these. Six major topics are considered in six chapters ensuing upon the first one that sums up the preliminaries.

The second chapter starts with the fundamental theorem of Minkowski and its various extensions and generalizations, among them Siegel's interesting observation that Minkowski's theorem is a special case of Parseval's formula for multiple Fourier series, and finally homogeneous and inhomogeneous minima of convex bodies with respect to a lattice is studied. In the third chapter families of lattices are considered, Mahler's selection theorem is proved, the role of the critical determinant is discussed, and questions of packings and converings of the plane are studied. Questions concerning the inhomogeneous determinant of a set are also touched upon.

The next two chapters deal with continuity properties connected with, and methods for reduction of, star bodies; methods for estimation from below of the critical and the inhomogeneous determinant are discussed. The last two chapters give a systematic study of arithmetical problems of the field. The absolute homogeneous minima of various forms are considered and applications to the theory of Diophantine approximation are exhibited; finally, questions related to the inhomogeneous minima of forms are investigated.

At the end of the book there is a vast bibliography that is complete for the period 1935—65.

*A. Máté* (Szeged)

**F. Klein, Elementarmathematik vom höheren Standpunkte aus.** Band 1: XII+309 Seiten, Band 2: XII+302 Seiten, Band 3: X+238 Seiten (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 14—16), 4. Auflage, Berlin, Verlag von Julius Springer, Nachdruck 1968.

Klein's weltberühmte Elementarmathematik hat eine epochemachende Bedeutung im Mathematiklehren. Der Inhalt der Bände ist der folgende. Band 1: Arithmetik, Algebra, Analysis; Band 2: Geometrie; Band 3: Präzisions- und Approximationsmathematik. Der gegenwertige Ausgabe ist ein Nachdruck von Fr. Seyfart's Umarbeitung (1933).

Ich zitiere die Worte des Buches über Bildungsreformen: „Die unabweisbare Notwendigkeit solcher Reformen liegt darin begründet, daß die diejenigen mathematischen Begriffsbildungen betreffen, die heutzutage die Anwendungen der Mathematik auf alle möglichen Gebiete durchaus beherrschen und ohne die alle Studien an der Hochschule, schon die einfachsten Studien über Experimentalphysik, gänzlich in der Luft schweben."

*J. Berkes* (Szeged)