Characterizations of regular varieties

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To Professor L. Rédei on his 70th birthday

Following A. I. MALCEV [5], an algebra A is called regular if every congruence on A is determined by anyone of its classes. A variety is regular if it contains only regular algebras. We shall prove that regular varieties may be characterized by means of identities as well as conditional identities. Similar results were obtained for varieties of algebras with distributive congruence lattices by B. JÓNSSON and for varieties with ideals by K. FICHTNER (see [4], resp. [2]). Our result was suggested by that of Fichtner.

Our terminology and notation are essentially those of [1]. We suppose that the algebras under consideration possess the same set of basic operations, which therefore will be suppressed in notation. The basic operations and the unit operators $\delta_n^{(i)}$ will be regarded as derived operations ([1], pp. 126, 145).

For any $a, b \in A$, where A is an arbitrary algebra, let $\theta_{a,b}$ denote the minimal congruence on A for which a and b are congruent. If φ is any homomorphism of A, then θ_φ denotes the natural congruence on A corresponding to φ. We shall write $A = \{M\}$ to mean that M is a system of generators of A.

The following well-known facts will be needed in the sequel:

(1) Let $A = \{a, b, c\}$. For any $d \in A$ there exists a derived ternary operation μ for which $d = abc\mu$ holds.

(2) Let $A = \{a, b, c\}$. For any translation τ of A there exists a derived quaternary operation ν such that $d\tau = dabc\nu$ for all $d \in A$.

(3) Suppose $a, b_\gamma (\gamma \in \Gamma), c, d$ are given elements of A. In order that the relation $c \equiv d(\bigcup_{\gamma \in \Gamma} \theta_{a,b_\gamma})$ hold it is necessary and sufficient that there exist a natural number k, and, for $i = 1, 2, \ldots, k$, elements $c_i \in A$ (with $c_0 = c$, $c_k = d$), indices $\gamma_i \in \Gamma$ and translations τ_i of A, such that either the relations $c_{i-1} = a\tau_i$ and $c_i = b_{\gamma_i}\tau_i$ or the relations $c_{i-1} = b_{\gamma_i}\tau_i$ and $c_i = a\tau_i$ are satisfied (see MALCEV [5]).

The element $a \in A$ will be called regular if every congruence on A is determined by the class containing a. Then obviously we have:

(4) Any algebra A is regular if and only if all elements of A are regular.

The following important fact was first observed by J. HASHIMOTO ([3], Lemma 2. 1):

(5) The element $a\in A$ is regular if and only if for each pair of different elements $c, d\in A$ there exists a finite set $B(\subseteqq A)$ such that $\theta_{c,d}=\bigcup_{b\in B}\theta_{a,b}$.

Now we are going to give the above-mentioned characterizations of regular varieties.

Theorem. *For any variety* $\mathfrak{A}$ *the following three propositions are equivalent:*

I. $\mathfrak{A}$ *is regular.*

II. *In* $\mathfrak{A}$ *there exist derived ternary operations* $\mu_1, \ldots, \mu_n$ *and derived* $2n+3$*-ary operations* $\lambda_1, \ldots, \lambda_k$ *such that the following identities are satisfied:*

(6) $xxz\mu_i=z \qquad (i=1, \ldots, n),$

(7) $x=(xyz\mu_1)\ldots(xyz\mu_n)z\ldots zxyz\lambda_1,$

(8) $z\ldots z(xyz\mu_1)\ldots(xyz\mu_n)xyz\lambda_{j-1}=(xyz\mu_1)\ldots(xyz\mu_n)z\ldots zxyz\lambda_j \qquad (j=2, \ldots, k),$

(9) $z\ldots z(xyz\mu_1)\ldots(xyz\mu_n)xyz\lambda_k=y.$

III. *In* $\mathfrak{A}$ *there exist derived ternary operations* $\mu_1, \ldots, \mu_n$ *such that the identities* (6) *and the conditional identity*

(10) $$xyz\mu_1=z\wedge\ldots\wedge xyz\mu_n=z\Rightarrow x=y$$

hold.

Proof. (I→II) Denote by F the $\mathfrak{A}$-free algebra freely generated by x, y, z. Since F is regular, there exists, on account of (4) and (5), $w_1, \ldots, w_n\in F$ such that

(11) $$\theta_{x,y}=\bigcup_{i=1}^{n}\theta_{z,w_i}.$$

By virtue of (1), there exist derived ternary operations μ_i in $\mathfrak{A}$ satisfying $w_i=xyz\mu_i$ $(i=1, \ldots, n)$. First we show that the μ_i fulfil (6). Indeed, let φ the endomorphism of F defined by equations $x\varphi=y\varphi=x$, $z\varphi=z$. For $i=1, \ldots, n$ we have $\theta_\varphi\geqq\theta_{x,y}\geqq\theta_{z,w_i}$. Thus $xxz\mu_i=(xyz\mu_i)\varphi=w_i\varphi=z\varphi=z$; hence (6) is an identity in $\mathfrak{A}$.

Furthermore, from (11) it follows $x\equiv y(\bigcup_{i=1}^{n}\theta_{z,w_i})$ and thus on the basis of (2) and (3) there exist $x_0=x, x_1, \ldots, x_k=y\in F$, derived quaternary operations $v_1, \ldots, v_k$ in $\mathfrak{A}$, and elements $w_{t_j}=xyz\mu_{t_j}\in F$ $(1\leqq t_j\leqq n;\ j=1, \ldots, k)$ satisfying for any j either

(12) $$x_{j-1}=zxyzv_j,\ x_j=(xyz\mu_{t_j})xyzv_j$$

or

(13) $$x_{j-1}=(xyz\mu_{t_j})xyzv_j,\ x_j=zxyzv_j.$$

Now define the derived operations $\lambda_1, \ldots, \lambda_k$ in the following way: Let

$$x_1 \ldots x_{2n+3}\lambda_j = x_1 \ldots x_{n+t_j-1}(x_{n+t_j}x_{2n+1}x_{2n+2}x_{2n+3}\nu_j)x_{n+t_j+1}\ldots x_{2n}\delta_{2n}^{(n+t_j)}$$

if (12) holds for j, and let

$$x_1 \ldots x_{2n+3}\lambda_j = x_1 \ldots x_{t_j-1}(x_{t_j}x_{2n+1}x_{2n+2}x_{2n+3}\nu_j)x_{t_j+1}\ldots x_{2n}\delta_{2n}^{(t_j)}$$

if (13) holds for j. Then one can immediately verify that $x, y, z \in F$ satisfy (7)—(9), whence it follows that (7)—(9) are identities in $\mathfrak{A}$.

(II$\to$III) It is sufficient to show that the derived operations $\mu_1, \ldots, \mu_n$ in II fulfil (10). Substituting z for $xyz\mu_i$ $(i=1, \ldots, n)$ in (7)—(9), we obtain $x = z\ldots zxyz\lambda_1 = z\ldots zxyz\lambda_2 = \ldots = z\ldots zxyz\lambda_k = y$. Hence the implication (10) is identically true in $\mathfrak{A}$.

(III$\to$I) Let $A \in \mathfrak{A}$ and $a, c, d \in A$. Taking into account (4) and (5) it is enough to prove the existence of a finite set $B(\subseteqq A)$ satisfying $\theta_{c,d} = \bigcup_{b \in B} \theta_{a,b}$. Let $B = \langle cda\mu_i | i=1, \ldots, n\rangle$. For all $1 \leqq i \leqq n$ we have $cda\mu_i = cca\mu_i = a\,(\theta_{c,d})$, whence $\theta_{a,cda\mu_i} \leqq \theta_{c,d}$ and so $\bigcup_{b \in B} \theta_{a,b} \leqq \theta_{c,d}$. To prove that here actually equality holds let us consider the factor algebra $\bar{A} = A/\bigcup_{b \in B} \theta_{a,b}$. For any $u \in A$, let $\bar{u}$ denote that element of $\bar{A}$ which contains u. For all $1 \leqq i \leqq n$ we have $\bar{a} = \overline{cda\mu_i} = \bar{c}\bar{d}\bar{a}\mu_i$. Since $\bar{A} \in \mathfrak{A}$, we can apply (10) which implies $\bar{c} = \bar{d}$. This means that $c \equiv d(\bigcup_{b \in B} \theta_{a,b})$, that is $\theta_{c,d} \leqq \bigcup_{b \in B} \theta_{a,b}$, completing the proof.

From this theorem it follows easily that varieties of groups, rings, modules, quasigroups, and Boolean algebras, are regular. For these familiar varieties we have always $n=1$; e.g., for groups $xyz\mu = xy^{-1}z$, for quasigroups $xyz\mu = x/(z\backslash y)$, and for Boolean algebras $xyz\mu = x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}z + xyz$.

Added in proof. Recently, some other characterizations of regular varieties have been given by G. GRÄTZER (*J. Comb. Theory*, **8** (1970), 334—342) and R. WILLE (*Kongruenzklassengeometrien*, Springer Lecture Notes, **113** (1970), p. 71.)

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