On certain classes of Σ-structures

By F. GÉCSEG in Winnipeg (Canada)*)

To Prof. L. Rédei on his 70th birthday

1. The class of all groups can be given as the class of all structures with an associative binary operation (multiplication) and constant 1 satisfying the sentence $\Phi \equiv (x)(\exists y)(xy = yx = 1)$. On every such group $\mathfrak{G}$ define an additional unary operation (invertation), and denote by $\overline{\mathfrak{G}}$ the resulted structure. It is clear that the correspondence $\mathfrak{G} \to \overline{\mathfrak{G}}$ as well as its inverse preserves substructures, homomorphism, and free structures.

Let Σ be a set of sentences in a first order language $L(\tau)$. One can raise the following question: Under what conditions can additional operations $f \in \bar{F} - F$ be defined on every Σ-structure $\mathfrak{A} = \langle A; F, R\rangle$ such that the correspondence $\mathfrak{A} = \langle A; F, R\rangle \to \overline{\mathfrak{A}} = \langle A; \bar{F}, R\rangle$ and its inverse have the three preserving properties above? As it was shown by G. GRÄTZER in [1], such additional operations can be defined if and only if Σ has the Inverse Preserving Property, $\mathscr{F}_\Sigma(\omega)$ exists and is strong. (The definitions are listed below.)

It is well known that $\{\overline{\mathfrak{G}} | \mathfrak{G} \in \Phi^*\}$ is a universal class. In this paper we show that, under the conditions given by Grätzer, every class $\{\overline{\mathfrak{A}} | \mathfrak{A} \in \Sigma^*\}$ is universal. This answers the Problem 86 of GRÄTZER raised in his book [1].

2. For the sake of completeness we recall some definitions from Chapter 8 of [1].

Let $\Phi \in \Sigma$ be of the following form:

$$(x_0)\dots(x_{n_0-1})(\exists y_0)(x_{n_0})\dots(x_{n_1-1})(\exists y_1)\dots(\exists y_k)(x_{n_k})\dots(x_{n-1})$$
$$\psi(x_0, \dots, x_{n_0-1}, y_0, x_{n_0}, \dots, x_{n_1-1}, y_1, \dots, y_k, x_{n_k}, \dots, x_{n-1}),$$

where $0 \leqq n_0 \leqq n_1 \cdots \leqq n_k \leqq n$; $0 = n_0$ means that no universal quantifier precedes $\exists y_0$, $n_0 = n_1$ means that there is no universal quantifier between $\exists y_0$ and $\exists y_k$, ψ contains no quantifiers. Set $e(\Phi) = k+1$. The concepts of *Φ-l inverse* and *Φ-l*

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sequence are defined for all $0 \leqq l \leqq e(\Phi)$ by induction on l. Let $\mathfrak{A}$ be a Σ-structure, $a_0, a_1, \ldots$ and $b_0, b_1, \ldots \in A$.

(i) b_0 is a Φ-0 inverse of $a_0, \ldots, a_{n_0-1}$ in $\mathfrak{A}$ if

$$(x_{n_0}) \ldots (x_{n_1-1})(\exists y_1) \ldots (\exists y_k)(x_{n_k}) \ldots (x_{n-1})$$

$$\psi(a_0, \ldots, a_{n_0-1}, b_0, x_{n_0}, \ldots, x_{n_1-1}, y_1, \ldots, y_k, x_{n_k}, \ldots, x_{n-1})$$

holds in $\mathfrak{A}$; in this case, $a_0, \ldots, a_{n_0-1}, b_0$ is a Φ-0 sequence;

(ii) b_l is a Φ-l inverse of $a_0, \ldots, a_{n_l-1}$ in $\mathfrak{A}$ if there exists a Φ-$(l-1)$ sequence $a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_{l-1}-1}, b_{l-1}$ such that

$$(x_{n_l}) \ldots (x_{n_{l+1}-1})(\exists y_{l+1}) \ldots (\exists y_k)(x_{n_k}) \ldots (x_{n-1})$$

$$\psi(a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_{l-1}-1}, b_{l-1}, a_{n_{l-1}}, \ldots, a_{n_l-1}, b_l,$$

$$x_{n_l}, \ldots, x_{n_{l+1}-1}, y_{l+1}, \ldots, y_k, x_{n_k}, \ldots, x_{n-1})$$

holds in $\mathfrak{A}$. Then, $a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_{l-1}-1}, b_{l-1}, a_{n_{l-1}}, \ldots, a_{n_l-1}, b_l$ is a Φ-l sequence.

Φ-inverse means Φ-l inverse for some $l < e(\Phi)$ and *Σ-inverse* means Φ-inverse for some $\Phi \in \Sigma$.

Let $\mathfrak{A}$ be a Σ-structure and let $\mathfrak{B}$ be a subtructure of $\mathfrak{A}$. Then $\mathfrak{B}$ is a *Σ-substructure* of $\mathfrak{A}$ if whenever $a_0, \ldots, a_t \in B$, $b \in A$ and b is a Σ-inverse of $a_0, \ldots, a_t$ in $\mathfrak{A}$, then $b \in B$.

Let n be a positive integer. The set $P_n(\Sigma)$ of *n-ary Σ-polynomial symbols* is defined by rules (i)—(iv) below.

(i) $x_i \in P_n(\Sigma)$, $i=0, \ldots, n-1$;

(ii) if $P_0, \ldots, P_{n_\gamma-1} \in P_n(\Sigma)$ then $f_\gamma(P_0, \ldots, P_{n_\gamma-1}) \in P_n(\Sigma)$ for $\gamma < 0_0(\tau)$;

(iii) if $\Phi \in \Sigma$, $l < e(\Phi)$, n_l universal quantifiers precede $\exists y_l$ and $P_0, \ldots, P_{n_l-1} \in P_n(\Sigma)$, then $\Phi^{(l)}(P_0, \ldots, P_{n_l-1}) \in P_n(\Sigma)$;

(iv) $P_n(\Sigma)$ is the smallest set satisfying (i)—(iii).

Let $P \in P_n(\Sigma)$, let $\mathfrak{A}$ be a Σ-structure, and let $a_0, \ldots, a_{n-1} \in A$. Then $P_{\mathfrak{A}}(a_0, \ldots, a_{n-1})$ (or simply $P(a_0, \ldots, a_{n-1})$) is a subset of A defined as follows:

(i) if $P = x_i$, then $P(a_0, \ldots, a_{n-1}) = \{a_i\}$;

(ii) if $P = f_\gamma(P_0, \ldots, P_{n_\gamma-1})$, then $P(a_0, \ldots, a_{n-1}) = \{a \mid a = f_\gamma(b_0, \ldots, b_{n_\gamma-1})$ for some $b_i \in P_i(a_0, \ldots, a_{n-1})$, $i=0, \ldots, n_\gamma-1\}$;

(iii) if $P = \Phi^{(l)}(P_0, \ldots, P_{n_l-1})$, then $P(a_0, \ldots, a_{n-1}) = \{a \mid a$ is *a* Φ-l inverse of some $b_0, \ldots, b_{n_l-1}$ with $b_i \in P_i(a_0, \ldots, a_{n-1})$, $i=0, \ldots, n_l-1\}$.

$P_{\mathfrak{A}}$ is called a *Σ-polynomial* over $\mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures and let φ be a mapping of A into B. Then φ is called a *Σ-homomorphism* if φ is a homomorphism, and if for any positive integer n, $P \in P_n(\Sigma)$ and $a_0, \ldots, a_{n-1} \in A$ we have

$$P(a_0, \ldots, a_{n-1})\varphi = P(a_0\varphi, \ldots, a_{n-1}\varphi).$$

Let $\mathfrak{B}$ be a Σ-substructure of the Σ-stucture $\mathfrak{A}$. Then $\mathfrak{B}$ is said to be a *slender* Σ-substructure if for any positive integer n, $P \in P_n(\Sigma)$ and $a_0, \ldots, a_{n-1} \in B$ we have that $P_{\mathfrak{A}}(a_0, \ldots, a_{n-1}) = P_{\mathfrak{B}}(a_0, \ldots, a_{n-1})$.

Let α be an ordinal. $\mathscr{F}_\Sigma(\alpha)$ is the *free Σ-structure* with α generators, if the following conditions are satisfied:

(i) $\mathscr{F}_\Sigma(\alpha)$ is a Σ-structure,

(ii) $\mathscr{F}_\Sigma(\alpha)$ is Σ-generated by the elements $x_0, \ldots, x_\gamma, \ldots$ $(\gamma < \alpha)$;

(iii) if $\mathfrak{A}$ is a Σ-structure and $a_0, \ldots, a_\gamma, \ldots \in A$ for $\gamma < \alpha$, then the mapping $\varphi\colon x_\gamma \to a_\gamma$, $\gamma < \alpha$ can be extended to a Σ-homomorphism $\bar{\varphi}$ of $\mathscr{F}_\Sigma(\alpha)$ into $\mathfrak{A}$.

A set Σ of sentences is said to have the *Inverse Preserving Property* (IP) if every Σ-substructure is slender.

Moreover, a free Σ-structure is called *strong* if the mapping $\bar{\varphi}$ given in the definition of the free Σ-stucture is always unique.

We note that if the free Σ-structure $\mathscr{F}_\Sigma(\omega)$ exists and is strong, then all free Σ-structures exist and all are strong (see Theorem 54.2 in [1]).

In addition to these definitions we introduce some notations.

3. Let $\bar{K}$ denote the equational class generated by the free structure $\overline{\mathscr{F}}_\Sigma(\omega)$ of Theorem 54.3 in [1] and let $\Sigma_{\bar{K}}$ be the set of all equations which hold in $\bar{K}$.

We introduce the notation Σ_R for the set of all sentences

$$(x_{0_1}) \ldots (x_{0_n}) \ldots (x_{m-1_1}) \ldots (x_{m-1_n}) r\big(p_0(x_{0_1}, \ldots, x_{0_n}), \ldots, p_{m-1}(x_{m-1_1}, \ldots, x_{m-1_n})\big)$$

which hold in $\overline{\mathscr{F}}_\Sigma(\omega)$, where $r \in R$ and $p_0, \ldots, p_{m-1}$ are polynomial symbols over $\bar{F}$.

As in Theorem 54.3 of [1] for every Σ-polynomial symbol $P(\in P_n(\Sigma))$ we define k_P n-ary operations $f_0^P, \ldots, f_{k_P-1}^P$ as follows:

Take $\mathscr{F}_\Sigma(n)$ with the Σ-generators $x_0, \ldots, x_{n-1}$; define $f_i^P(x_0, \ldots, x_{n-1})$, $i < k_P$ such that

$$P(x_0, \ldots, x_{n-1}) = \{f_i^P(x_0, \ldots, x_{n-1}) \mid i < k_P\};$$

let $\mathfrak{A}$ be an arbitrary Σ-structure, $a_0, \ldots, a_{n-1} \in A$ and φ a Σ-homomorphism of $\mathscr{F}_\Sigma(n)$ into $\mathfrak{A}$ with $x_0\varphi = a_0, \ldots, x_{n-1}\varphi = a_{n-1}$. Let

$$f_i^P(a_0, \ldots, a_{n-1}) = f_i^P(x_0, \ldots, x_{n-1})\varphi \qquad (i = 0, \ldots, k_P - 1).$$

Let $P \in P_n(\Sigma)$. Then there exists a formula $r_P(x_0, \ldots, x_{n-1}, y)$ in $L(\tau)$ such that if $\mathfrak{A}$ is a Σ-structure and $a_0, \ldots, a_{n-1}, b \in A$, then $b \in P(a_0, \ldots, a_{n-1})$ if and only if $r_P(a_0, \ldots, a_{n-1}, b)$ (see [1], Lemma 49.6).

Now we define for every $P \in P_n(\Sigma)$ a sentence Φ_P in the following manner:

$$\Phi_P \equiv (x_0)\dots(x_{n-1})(y)\Big(r_P(x_0,\dots,x_{n-1},f_0^P(x_0,\dots,x_{n-1}))\wedge\dots$$
$$\dots\wedge r_P(x_0,\dots,x_{n-1},f_{k_P-1}^P(x_0,\dots,x_{n-1}))\wedge\big(r_P(x_0,\dots,x_{n-1},y)\rightarrow$$
$$\rightarrow(y=f_0^P(x_0,\dots,x_{n-1})\vee\dots\vee y=f_{k_P-1}^P(x_0,\dots,x_{n-1}))\big)\Big).$$

Let $\Sigma_0=\{\Phi_P|\ P$ is a Σ-polynomial symbol$\}$.

Moreover, denote by Σ_U the set of all universal sentences from Σ.

Now we are ready to prove the following

Theorem. *If Σ has the Inverse Preserving Property and $\mathcal{F}_\Sigma(\omega)$ exists and is strong then $K=\{\overline{\mathfrak{A}}|\mathfrak{A}\in\overline{\Sigma}^*\}$ is a universal class.*

Proof. The class K is closed with respect to substructures (see [1], Theorem 54.3). We shall prove that K is a universal class by showing that

$$K=(\Sigma_K\cup\Sigma_R\cup\Sigma_0\cup\Sigma_U)^*,$$

i.e. K is an axiomatic class; see [1], Corollary to Theorem 43.3 for this characterization of universal classes.

First we show that $\overline{\mathfrak{A}}\in(\Sigma_K\cup\Sigma_R\cup\Sigma_0\cup\Sigma_U)^*$ if $\mathfrak{A}\in\Sigma^*$.

$\overline{\mathcal{F}}_\Sigma(\alpha)$ is a free structure in K of Theorem 54.3 of [1] so $\overline{\mathfrak{A}}$ is a homomorphic image of $\overline{\mathcal{F}}_\Sigma(\alpha)$ for some α, i.e. Σ_K and Σ_R hold in $\overline{\mathfrak{A}}$.

Let $P\in P_n(\Sigma)$ be a Σ-polynomial symbol and take $\overline{\mathcal{F}}_\Sigma(n)$ with the generators $x_0,\dots,x_{n-1}$. Let $y\in P(x_0,\dots,x_{n-1})$ be arbitrary. By the definition of the operations $f_i^P(x_0,\dots,x_{n-1})$ $(i<k_P)$ and the formulas $r_P(x_0,\dots,x_{n-1},y)$ it can easily be seen that $r_P(x_0,\dots,x_{n-1},f_i^P(x_0,\dots,x_{n-1}))$ holds and there exists a $j\ (<k_P)$ such that $y=f_j^P(x_0,\dots,x_{n-1})$. Now take arbitrary $a_0,\dots,a_{n-1}\in A$. Then the mapping $x_0\varphi=a_0,\dots,x_{n-1}\varphi=a_{n-1}$ can be extended to a homomorphism $\bar\varphi$ of $\overline{\mathcal{F}}_\Sigma(n)$ into $\overline{\mathfrak{A}}$ such that $\bar\varphi$ is a Σ-homomorphism of $\mathcal{F}_\Sigma(n)$ into $\mathfrak{A}$. So, by the definition of the Σ-homomorphism and operations $f_i^P\,(i<k_P)$, $r_P\,(a_0,\dots,a_{n-1},f_i^P(a_0,\dots,a_{n-1}))$ also holds. It can be shown in the same way that if $r_P(a_0,\dots,a_{n-1},b)$ holds, i.e. $b\in P(a_0,\dots,a_{n-1})$, then there exists a $j(<k_P)$ such that $b=f_j^P(a_0,\dots,a_{n-1})$. But P and $a_0,\dots,a_{n-1},b\in A$ are arbitrary, so Σ_0 holds in $\overline{\mathfrak{A}}$.

$\overline{\mathfrak{A}}\in\Sigma_U^*$ is obviously valid.

Conversely, let $\hat{\mathfrak{B}}=(B;\overline{F},R)\in(\Sigma_K\cup\Sigma_R\cup\Sigma_0\cup\Sigma_U)^*$. We have to show that $\hat{\mathfrak{B}}\in\{\overline{\mathfrak{A}}|\mathfrak{A}\in\Sigma^*\}$. Denote by $\mathfrak{B}$ the (F,R)-reduct of $\hat{\mathfrak{B}}$.

$\hat{\mathfrak{B}}\in\Sigma_U^*$ implies that for every $b_0,\dots,b_{n-1}\in B$ and $P\in P_n(\Sigma)$, $r_P\,(b_0,\dots,b_{n-1}, f_i^P(b_0,\dots,b_{n-1}))$ $(i<k_P)$ holds, i.e. $|P\,(b_0,\dots,b_{n-1})|\geqq 1$ which means that every $\Phi\in\Sigma$ having at least one existential quantifier holds in $\mathfrak{B}$. Since $\hat{\mathfrak{B}}\in\Sigma_U^*$ thus every universal sentence $\Phi\in\Sigma$ also holds in $\mathfrak{B}$. Therefore, $\mathfrak{B}$ is a Σ-structure.

Take an arbitrary generating system $\langle b_0, \ldots, b_\gamma, \ldots, \rangle_{\gamma<\alpha}$ of $\hat{\mathfrak{B}}$. Because $\overline{\mathscr{F}}_\Sigma(\alpha)$ is free in $\bar{K}$ and $\hat{\mathfrak{B}} \in \Sigma_{\bar{K}}^*$, every mapping $\varphi: x_\gamma \to b_\gamma$ $(\gamma<\alpha)$ can be extended to an algebra-homomorphism $\bar{\varphi}$ of $\overline{\mathscr{F}}_\Sigma(\alpha)$ onto $\hat{\mathfrak{B}}$. (A mapping $\psi: \mathfrak{A}=(A; F, R) \to \mathfrak{A}'=(A'; F, R)$ is said to be an algebra-homomorphism if $f(a_0, \ldots, a_{n-1})\psi = f(a_0\psi, \ldots, a_{n-1}\psi)$ for every $f \in F$ and $a_0, \ldots, a_{n-1} \in A$.)

Furthermore, if

$$r\big(p_0(x_{0_1}, \ldots, x_{0_n}), \ldots, p_{m-1}(x_{m-1_1}, \ldots, x_{m-1_n})\big)$$

holds in $\overline{\mathscr{F}}_\Sigma(\alpha)$, then

$$(x_{0_1}) \ldots (x_{0_n}) \ldots (x_{m-1_1}) \ldots (x_{m-1_n})\, r\big(p_0(x_{0_1}, \ldots, x_{0_n}), \ldots, p_{m-1}(x_{m-1_1}, \ldots, x_{m-1_n})\big) \in \Sigma_R,$$

because every mapping of the generating system of $\overline{\mathscr{F}}_\Sigma(\alpha)$ into $\overline{\mathscr{F}}_\Sigma(\omega)$ can be extended to a homomorphism of $\overline{\mathscr{F}}_\Sigma(\alpha)$ into $\overline{\mathscr{F}}_\Sigma(\omega)$. But $\hat{\mathfrak{B}} \in \Sigma_R^*$, so φ is a homomorphism of $\overline{\mathscr{F}}_\Sigma(\alpha)$ onto $\hat{\mathfrak{B}}$.

It remains to prove that $\bar{\varphi}$ is a Σ-homomorphism of $\mathscr{F}_\Sigma(\alpha)$ onto $\mathfrak{B}$. For this, by the definition of Σ-homomorphism, it is enough to show that

$$P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})\bar{\varphi} = P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi})\big(=P(b_{\gamma_0}, \ldots, b_{\gamma_{n-1}})\big)$$

for every $P \in P_n(\Sigma)$ and $\gamma_0, \ldots, \gamma_{n-1} < \alpha$. Take $y \in P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})$. Then there exists an f_i^P $(i<k_P)$ such that $y = f_i^P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})$. So $y\bar{\varphi} = f_i^P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi})$. But $\hat{\mathfrak{B}} \in \Sigma_0^*$, thus $r_P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi}, f_i^P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi}))$ holds, i.e. $y\bar{\varphi} \in P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi})$. Conversely, take $b \in P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi})$, i.e. $r_P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi}, b)$ holds. Because $\hat{\mathfrak{B}} \in \Sigma_0^*$, thus there exists an f_j^P $(j<k_P)$ such that $f_j^P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi}) = b$. It is clear $f_j^P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})\bar{\varphi} = b$ and $f_j^P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}}) \in P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})$. These prove that $P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})\bar{\varphi} = P(x_{\gamma_0}\bar{\varphi}, \ldots, x_{\gamma_{n-1}}\bar{\varphi})$.

It is clear, by Theorem 54.3 of [1], that taking the correspondence $\mathfrak{B} \to \bar{\mathfrak{B}}$ given in Theorem 54.3 of [1], we have $\bar{\mathfrak{B}} = \hat{\mathfrak{B}}$. This ends the proof of the Theorem.

Reference

[1] G. GRÄTZER, *Universal Algebra* (Princeton, N. J., 1968).

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