Characterization of completely regular inverse semigroups

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To Professor L. Rédei on his seventieth birthday

Let S be a semigroup[1]). Following E. S. LJAPIN we shall say that S is a completely regular semigroup if to any element a in S there exists at least one element x in S such that $axa = a$ and $ax = xa$. S is said to be an inverse semigroup (or generalized group) if to each element a of S there exists a unique element $x \in S$ so that $axa = a$ and $xax = x$. (See V. V. VAGNER [8] and G. B. PRESTON [7].) Professors A. H. CLIFFORD and G. B. PRESTON write in their monograph: "Inverse semigroups constitute probably the most promising class of semigroups for study at the present time, since they are not too far away from groups." An important result due to VAGNER establishes that every inverse semigroup is isomorphic to a semigroup of one-to-one partial transformations. A complete description of inverse semigroups which are unions of groups is due to A. H. CLIFFORD [1]. It is also known that a semigroup S is completely regular if and only if it is a union of groups. In this note we shall give some ideal-theoretical characterizations of completely regular inverse semigroups. Our first criterion reads as follows.

Theorem 1. *A semigroup S is a completely regular inverse semigroup if and only if the equality*

$$L \cap R = LR \tag{1}$$

holds for any left ideal L and for any right ideal R of S.

Proof. *Necessity.* Let S be a completely regular inverse semigroup. Then the relation

$$L \cap R = RL \tag{2}$$

holds for each left ideal L and for each right ideal R of S, because S is a regular semigroup. Now the exercise 4. 2. 2 in [2] implies that every one-sided (left or right)

[1]) For the notation and terminology we refer to [2].

ideal of S at the same time is a two-sided ideal. Therefore the equality (1) is a consequence of (2).

Sufficiency. Suppose that S is a semigroup satisfying the equation (1) for any left ideal L and for any right ideal R of S. Then (1) implies that

$$S \cap aS = SaS \tag{3}$$

and

$$Sa \cap S = SaS \tag{4}$$

holds for each element a in S. The relations (3), (4) imply that S is a centric semigroup. We show that every one-sided ideal of S is a two-sided ideal. Applying (1) in case $R=S$ we have

$$L \cap S = LS \tag{5}$$

for any left ideal L of S. This means that every left ideal L of S is a two-sided ideal of S. Similarly in case $L = S$ (1) implies that

$$S \cap R = SR \tag{6}$$

for any right ideal R of S. Hence we conclude that any right ideal R of S at the same time is a two-sided ideal of S.

Next we show that S is regular. Let a be an arbitrary element of S. Then the principal left ideal $(a)_L$ generated by a coincides with the principal two-sided ideal (a) of S. Applying (1) in case $L=R=(a)$ we obtain that

$$(a) \cap (a) = (a)(a) \tag{7}$$

for any element a in S. In view of (3) and (4) the relation (7) implies that

$$a \in a^2 \cup aSa, \tag{8}$$

that is a is a regular element of S. Since the idempotent elements of a centric semigroup lie in the center (see [2], vol. II, p. 197) we conclude that the idempotent elements of S commute. Therefore, by a well-known characterization of the inverse semigroups (see [2], vol. I, p. 28) S is an inverse semigroup. Finally from exercise 4. 2. 2 in [2] it follows that if S is an inverse semigroup every one-sided ideal of which is two-sided then S is a union of groups. This implies that S is a completely regular semigroup.

The proof of Theorem 1 is complete.

Theorem 2. *The following conditions on a semigroup S are equivalent:*

(A) *S is a completely regular inverse semigroup.*

(B) *$L \cap R = LR$ for any left ideal L and for any right ideal R of S.*

(C) $L_1 \cap L_2 = L_1 L_2$ *and* $R_1 \cap R_2 = R_1 R_2$ *for any two left ideals* L_1, L_2 *and for any two right ideals* R_1, R_2 *of* S, *respectively.*

(D) S *is a semilattice of groups.*

Proof. The statement of Theorem 2 follows at once from Theorem 3 of the author's paper [4] and from Theorem 1 of this note. For other related results see [3] and [5].

Bibliography

[1] A. H. Clifford, Semigroups admitting relative inverses, *Ann. of Math.*, **42** (1941), 1037—1049.

[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups.* I—II (Providence, R. I., 1961; 1967).

[3] S. Lajos, A note on completely regular semigroups, *Acta Sci. Math.*, **28** (1967), 261—265.

[4] S. Lajos, A csoportok félhálójaként előállítható félcsoportok ideálelméleti jellemzése, *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.*, **19** (1969), 113—115.

[5] S. Lajos, Note on semigroups which are semilattices of groups, *Proc. Japan Acad.*, **44** (1968), 805—806.

[6] Е. С. Ляпин, *Полугруппы* (Москва, 1960).

[7] G. B. Preston, Inverse semi-groups, *J. London Math. Soc.*, **29** (1954), 396—403.

[8] В. В. Вагнер, Обобщенные группы, *Докл. Акад. Наук СССР*, **84** (1952), 1119—1122.

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