A generalization of ordered algebraic systems

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To Professor L. Rédei on his 70th birthday

Introduction

The investigation of partially ordered algebraic systems dates back to that of fully ordered fields. One gets from the full order to the partial one by disregarding the trichotomy of the relation. In this paper the full order will be weakened in another way.

The trichotomy, the antisymmetry and the transitivity of a relation give, already, the full order of the different elements. One can suppose in this case reflexivity, too, practically without any loss of generality. In fact, a little reduction of a full order always has to be reflexive. Therefore, we shall deal with reflexive relations only. We shall consider reflexive relations on sets, groups, and fields.

1. Reflexive relations on a set

Let I_2 denote the two-element Boolean algebra with the lower bound 0 and the upper bound 1. We shall consider these elements as natural numbers, too. A relation R on a set $\mathcal{S}$ will be regarded as a mapping

$$R: \mathcal{S} \times \mathcal{S} \to I_2$$

of the Cartesian product of two copies of $\mathcal{S}$ into I_2. The relation R on $\mathcal{S}$ is called *reflexive* if $R(a, a) = 1$ holds for each $a \in \mathcal{S}$.

The reflexive relations on $\mathcal{S}$ form a complete Boolean algebra with the lattice operations

$$S = \bigwedge_{\nu} R_{\nu} \quad \text{and} \quad R = \bigvee_{\nu} R_{\nu}$$

where

$$S(a, b) = \min \big(R_{\nu}(a, b)\big) \quad \text{and} \quad R(a, b) = \max \big(R_{\nu}(a, b)\big).$$

The complement R' of R is defined in the following manner: $R'(a, b) = 1-R(a, b)$ if $a \neq b$ $(a, b \in \mathcal{S})$, and $R'(a, a)=1$ for any $a \in \mathcal{S}$.

Instead of $R(a, b)=1$ one usually says that the relation $R(a, b)$ is valid.

We shall consider the following properties of a relation.

1) Antisymmetry, which means that the relations $R(a, b)$ and $R(b, a)$ are not valid simultaneously, providing $a \neq b$.

2) Trichotomy, which means that exactly one of the following relations: $a=b$; $a \neq b$ and $R(a, b)$; $a \neq b$ and $R(b, a)$ is valid.

3) Transitivity, which means that the validity of $R(a, b)$ and $R(b, c)$ implies $R(a, c)$.

4) A property, which can be called the dual to transitivity, will be referred to as layerity; this means that from the validity of $R(a, b)$ follows $R(a, c)$ or $R(c, a)$, for any c.

The considerations are much easier if these properties are required only for different elements. At the same time their meanings change only slightly. We shall identify these new properties not by numbers, but by matrices.

Let a, b and c be different elements of a given set $\mathcal{S}$, and, further, let R be a reflexive relation on $\mathcal{S}$. We shall say that R has the property:

$\binom{10}{00}$ *Antisymmetry* if and only if $R(a, b)+R(b, a) \leqq 1$ holds,

$\binom{01}{00}$ *Trichotomy* if and only if $R(a, b)+R(b, a) \geqq 1$ holds,

$\binom{00}{10}$ *Transitivity* if and only if $R(a, b)=R(b, c)=1$ implies $R(a, c)=1$,

$\binom{00}{01}$ *Layerity* if and only if $R(a, b)=R(b, c)=0$ implies $R(a, c)=0$.

We shall use the sum of the suitable matrices for denoting the properties of a relation. Thus, R is an $\binom{ab}{cd}$-relation (where the elements of this matrix are either 0 or 1) if and only if R has those properties out of $\begin{pmatrix}\text{Antisymmetry} & \text{Trichotomy}\\ \text{Transitivity} & \text{Layerity}\end{pmatrix}$ in whose place a number 1 stands.

One can immediately see:

Proposition 1. *R' is an $\binom{ab}{cd}$-relation if and only if R is a $\binom{ba}{dc}$-relation.*

Proposition 2. *If each R_ν is an $\binom{a0}{c0}$-relation then so is $R=\bigvee_\nu R_\nu$.*

Proposition 3. *If each S_ν is a $\binom{0b}{0d}$-relation then so is $S=\bigvee_\nu S_\nu$.*

Though a full order has all these properties, one never assumes layerity. This shows that this property is a consequence of the other ones. More generally:

Theorem 1. *Each $\binom{10}{01}$-relation is a $\binom{10}{11}$-relation, and each $\binom{01}{10}$-relation is a $\binom{01}{11}$-relation.*

Proof. It is sufficient, by Proposition 1, to prove the first part only. Let the different elements a, b, and c of $\mathcal{S}$ be so chosen that $R(b, c)=1$ and $R(a, c)=0$

hold. Then we get, using antisymmetry, $R(c, b)=0$, and, invoking layerity, $R(a, b)=0$; i.e. $R(a, b)=R(b, c)=1$ is possible only in case $R(a, c)=1$.

Using Szpilrajn's result, which states that each partial order is an intersection of full orders (see [3]), one obtains

Theorem 2. *Any* $\binom{01}{01}$*-relation is a union of full orders.*

Proof. Let R be a $\binom{01}{01}$-relation. It follows, from Proposition 1, that R' is a partial order. Szpilrajn's result and Propositions 1 and 3 complete the proof.

We define the rank of an $\binom{ab}{cd}$-relation R by $r(R) = a+b+c+d$.

Two special cases of the $\binom{ab}{cd}$-relations are familiar, namely the partial orders, which are the $\binom{10}{10}$-relations, and the tournaments, i.e. the $\binom{11}{00}$-relations. Both of them have rank two and the full order has rank four. So, it seems to be useful to consider all types of relations whose ranks are at least two.

1. $r(R)=4$ is equivalent for R to be a full order.

2. $r(R)=3$ is possible, by Theorem 1, if and only if R is either a $\binom{10}{11}$-relation or a $\binom{01}{11}$-relation. These are complements to each other. The first one will be called, as a special partial order, *layerwise order* and the second one will be called an *overorder* since it is, by Theorem 2, an "extension" of some full order.

3. $r(R)=2$ is possible, also by Theorem 1, in four cases. A $\binom{11}{00}$-relation is a *tournament*, the complement of which is also a tournament. The other relation whose complement has the same type is the $\binom{00}{11}$-relation, which will be called a *layer* relation. Finally, a partial order is a $\binom{10}{10}$-relation, and its complement is a $\binom{01}{01}$-relation.

There are two trivial cases, namely the two bounds of the Boolean algebra, of the reflexive relations. One of them is the *trivial order* which maps each pair of distinct elements to 0. The other relation which sends each pair to 1 will be called *full overorder*.

The considerations of tournaments and partially ordered sets are familiar, while the consideration of $\binom{01}{01}$-relation is dual to that of the partially ordered sets.

Now, we are going to describe the layer-relations, especially the overorders and the layerwise orders.

Let $\mathscr{S}$ be a set with a reflexive relation R and to each element x of $\mathscr{S}$ let there be given a set $\mathscr{S}_x$ with a reflexive relation R_x. Let further $\underset{x\in\mathscr{S}}{J}\mathscr{S}_x$ denote the set of pairs (a, x) with $x\in\mathscr{S}$ and $a\in\mathscr{S}_x$ endowed with the relation R^* for which

$$R^*((a, x), (b, y))=1$$

if and only if

$$\text{either } x\neq y \text{ and } R(x, y)=1 \text{ or } x=y \text{ and } R_x(a, b)=1.$$

$\underset{x\in\mathscr{S}}{J}\mathscr{S}_x$ will be called the join of $\mathscr{S}_x$ over $\mathscr{S}$.

The set $\mathcal{S}$ with the relation R and the set $\mathcal{S}'$ with the relation R' will be called *similar* (to each other) if there exists a one-to-one mapping Φ: $\mathcal{S} \to \mathcal{S}'$ such that $R'(\Phi(a), \Phi(b)) = R(a, b)$ for every $a, b \in \mathcal{S}$.

Theorem 3. *If $\mathcal{S}$ is a set with a layer R on it, then it is similar to a join of sets, which are either trivially ordered sets or fully overordered ones, over a fully ordered set.*

Proof. The theorem is obvious if $\mathcal{S}$ has no more than two elements. Therefore, we suppose that $\mathcal{S}$ has at least three elements. Let the distinct elements a and b of $\mathcal{S}$ be chosen so that $R(a, b) = R(b, a)$, and let $x \in \mathcal{S}$ be different from both a and b. From $R(a, b) = R(b, x)$ it follows $R(a, b) = R(a, x)$, which means that either both of $R(a, x)$ and $R(b, x)$ are equal to $R(a, b)$ or both of them are different from it. A similar result holds for $R(x, a)$ and $R(x, b)$.

Now, let a, b and c be different elements of $\mathcal{S}$ such that $R(a, b) = R(b, a)$ and $R(b, c) = R(c, b)$ are valid. Then it follows that $R(b, c) = R(a, c)$ and $R(c, b) = R(c, a)$, and similarly that $R(b, a) = R(c, a)$ and $R(a, b) = R(a, c)$. That is, these values are equal to each other. Thus, the relation „$S(a, b) = 1$ *if and only if* $R(a, b) = R(b, a)$" is such an equivalence that from $S(a, b) = S(c, d) = 1$ follows $R(a, c) = R(b, d)$.

Let $\mathcal{S}_x$ denote the set of elements of $\mathcal{S}$ for which $S(a, x) = 1$ holds. Let, further, $\overline{\mathcal{S}}$ be the set of all sets $\mathcal{S}_x$.

As was proved, R is on each $\mathcal{S}_x$ either a trivial order or a full overorder. Further, the relation $\bar{R}(\mathcal{S}_x, \mathcal{S}_y) = R(x, y)$ does not depend on x and y and gives a full order of $\mathcal{S}$.

One can easily see that $\mathcal{S}$ is similar to $\underset{\bar{x} \in \overline{\mathcal{S}}}{J} \mathcal{S}_{\bar{x}}$ (where $\bar{x}$ denotes the set $\mathcal{S}_x$ regarded as an element of $\overline{\mathcal{S}}$ and $\mathcal{S}_{\bar{x}} = \mathcal{S}_x$) with the one-to-one mapping $x \to (x, \mathcal{S}_{\bar{x}})$.

Theorem 4. *Each overordered (layerwise ordered) set is similar to a join of fully overordered (trivially ordered) sets over a fully ordered set.*

Proof. Let R be an overorder on the set $\mathcal{S}$. Then for $a \neq b$ (both in $\mathcal{S}$) $R(a, b) = R(b, a)$ means, in view $R(a, b) + R(b, a) \geqq 1$, that $R(a, b) = 1$. Thus, no subset $\mathcal{S}_x$, given in Theorem 3, is trivially ordered. A similar consideration can be used to establish the case of layerwise order.

Theorem 3 shows that the maximal fully overordered and trivially ordered subsets of a set with a layer relation on it are indeed situated in layers.

2. Reflexive relations on a group

One always means by a *relation on a group* $\mathcal{G}$ an isotone relation, i.e. a relation R for which

$$R(ax, bx) = R(ya, yb) = R(a, b)$$

hold, for any elements a, b, x, y of $\mathcal{G}$.

Therefore a relation on a group is completely defined by the set $\mathcal{P}_R$ containing those elements x for which $R(e, x) = 1$, where e marks the unity of the group $\mathcal{G}$. $\mathcal{P}_R$ must be a set invariant under two-sided isotonity, which means $a^{-1}\mathcal{P}_R a \subseteq \mathcal{P}_R$ for each a in $\mathcal{G}$. The reflexivity of R is equivalent to $e \in \mathcal{P}_R$. Both $\mathcal{P}_{R'} \cap \mathcal{P}_R = \{e\}$ and $\mathcal{P}_{R'} \cup \mathcal{P}_R = \mathcal{G}$ are valid, i.e. $\mathcal{P}_{R'}$ and $\mathcal{P}_R$ uniquely determine each other. The elements of the *positive cone* $\mathcal{P}_R$ will be called *positive*.

Now, we are going to study the reflection of the given properties of relation R in the positive cone $\mathcal{P}_R$.

It is only a matter of routine to prove

Propositions 4, 5, 6, 7:

R is antisymmetric if and only if $\mathcal{P}_R \cap \mathcal{P}_R^{-1} = \{e\}$.

R is trichotomic if and only if $\mathcal{P}_R \cup \mathcal{P}_R^{-1} = \mathcal{G}$.

R is transitive if and only if $\mathcal{P}_R$ is a semigroup, i.e. if it is closed under the multiplication in $\mathcal{G}$.

R is layered if and only if $\mathcal{P}_R$ is prime, which means that no product of two elements of the complement of $\mathcal{P}_R$ belongs to the positive cone unless this product equals e.

We shall first consider the tournaments on groups. Similarly to the situation in case of a full order, one cannot define tournaments on every group.

Theorem 5. *A necessary and sufficient condition in order that there exists a tournament on a group $\mathcal{G}$ is that the inverse of any element $a \neq e$ of $\mathcal{G}$ be not conjugate to a.*

Proof. Let $[a]$ denote the set of conjugates of an element a of the group $\mathcal{G}$. If R is a tournament on $\mathcal{G}$ then of the elements a and a^{-1} one belongs to $\mathcal{P}_R$ and the other to $\mathcal{P}_R^{-1}$. Thus, using the normality of $\mathcal{P}_R$, we see that the set $[a]$ is contained either in $\mathcal{P}_R$ or in $\mathcal{P}_R^{-1}$; namely in that which does not contain $a^{-1} (a \neq e)$.

Conversely, let us choose just one of the sets $[a]$ and $[a^{-1}]$ for each $a \in \mathcal{G}$. Let $\mathcal{P}$ consist of the elements of the sets chosen. $\mathcal{P} \cup \mathcal{P}^{-1} = \mathcal{G}$ is obvious, while $\mathcal{P} \cap \mathcal{P}^{-1} = \{e\}$ easily follows from the disjointness of the sets $[a]$ and $[a^{-1}]$ for $a \neq e$.

Corollary 1. *One can define a tournament on a group if and only if different elements have different squares, and on a commutative group if and only if it has no elements of order two.*

Proof. One can write the equalities $a^{-1}=x^{-1}ax$ and $a=e$ in the forms $(ax)^2=x^2$ and $(ax)=x$, respectively. Theorem 5 proves the first assertion and the second one is a trivial consequence of it.

Corollary 2. *One can define a tournament on a group only if the group has no element of order two. For torsion groups this condition is also sufficient.*

Proof. (By L. RÉDEI) The necessity is, by Corollary 1, obvious. If the order of each element of a torsion group is odd then to each pair x and y of elements of this group there exists an odd number n such that $x^n=y^n=e$ hold. Thus, $x^2=y^2$ entails

$$x = x^{n+1} = (x^2)^{\frac{n+1}{2}} = (y^2)^{\frac{n+1}{2}} = y^{n+1} = y,$$

proving sufficiency.

Next, we shall consider layers on a group.

Theorem 6. *A layer on a group is either an overorder or a layerwise order.*

Proof. Let R be a layer on the group $\mathcal{G}$. Then both $\mathcal{P}=\mathcal{P}_R$ and $\mathcal{Q}=\mathcal{P}_{R'}$ are normal subsemigroups since R' is also a layer on $\mathcal{G}$. Assume for $a\in\mathcal{P}$ and $b\in\mathcal{Q}$ that $a^{-1}\in\mathcal{P}$ and $b^{-1}\in\mathcal{Q}$ are both valid. $\mathcal{P}\cup\mathcal{Q}=\mathcal{G}$ entails that ab belongs to one of these sets, e.g. to $\mathcal{P}$. So, it follows from $b=a^{-1}(ab)\in\mathcal{P}$ and from $\mathcal{P}\cap\mathcal{Q}=\{e\}$ that b is equal to e. Thus, we have either $\mathcal{P}^{-1}\subseteqq\mathcal{Q}$ or $\mathcal{Q}^{-1}\subseteqq\mathcal{P}$. Hence either $\mathcal{P}\cap\mathcal{P}^{-1}=\{e\}$ or $\mathcal{P}\cup\mathcal{P}^{-1}=\mathcal{G}$ is valid.

One can define, in such a manner as for ordered groups, the lexicographic product of two groups with reflexive relations. One can easily verify that a reflexive relation on a group induces both on a normal subgroup and on the factorgroup by the given normal subgroup a reflexive relation such that the automorphisms of the normal subgroup, induced by the inner ones of the group, send the positive cone of the normal subgroup into itself.

Now, let $\mathcal{N}$ be a normal subgroup of the group $\mathcal{G}$. Further, let there be given a reflexive relation R on $\mathcal{N}$ and a reflexive relation $\bar{R}$ on $\mathcal{G}/\mathcal{N}$. Let $\mathcal{P}$ consist of all elements of $\mathcal{G}$ for which

$$\text{either } a\in\mathcal{P}_R \quad \text{or} \quad a\notin\mathcal{N} \quad \text{and} \quad a\mathcal{N}\in\mathcal{P}_{\bar{R}}$$

is valid.

It is easy to verify that $\mathcal{P}$ is the positive cone of a reflexive relation on $\mathcal{G}$ if and only if $\mathcal{P}_R$ is sent by any inner automorphism of $\mathcal{G}$ into itself. (This is valid especially in the cases $\mathcal{P}_R=\{e\}$ and $\mathcal{P}_R=\mathcal{G}$.) In this case $\mathcal{G}$ will be called the *lexicographical extension* of $\mathcal{N}$ by $\mathcal{G}/\mathcal{N}$.

Theorem 7. *Each overordered group is a lexicographical extension of a fully overordered group by a fully ordered one. Any lexicographical extension of a fully overordered group by a fully ordered one is an overordered group.*

Proof. Denote by $\mathscr{P}$ the positive cone of an overorder on the group $\mathscr{G}$. The induced relation the normal subgroup $\mathscr{N} = \mathscr{P} \cap \mathscr{P}^{-1}$ is, obviously, a full overorder, and from Theorem 3 it follows that the induced relation on $\mathscr{G}/\mathscr{N}$ is a full order which is clearly compatible with multiplication. The second assertion is obtained by the special choice $\mathscr{P}_R = \mathscr{G}$.

One can easily prove, by using the complementary relation, that

Theorem 7a. *Each layerwise ordered group is a lexicographical extension of a trivially ordered group by a fully ordered one. Any lexicographical extension of a trivially ordered group by a fully ordered one is a layerwise ordered group.*

It follows directly from the definition that

Corollary *Those elements of a layerwise ordered group which are neither positive nor negative (i.e. inverses of positive ones) are pseudoidentities* (see [1]).

3. Reflexive relations on a field

The relations R and R' defined on the underlying set of a field are equivalent in that sense that the addition or multiplication with an element is compatible with the first relation if and only if it so is with the second one. So for the sake of convenience the unit element will be assumed to belong to the positive cone.

Proposition 8. *Let R be a reflexive relation on the additive group $\mathscr{F}^+$ of the field $\mathscr{F}$. If the multiplication with an element of this field is isotone (antitone) then this element (the negative of this element) belongs to the positive cone.*

Proof. The first assertion follows from $R(0, a) = R(0, 1)$ and the second one from $R(0, -a) = R(-1, 0) = R(0, 1)$.

From the compatibility of the multiplication with a reflexive relation on the additive group of a field follows, by Proposition 8, that the relation is a full overorder. Therefore we shall require the isotonity only for the positive elements. This means that the positive cone must be closed under multiplication. Conversely, if $\mathscr{P}$ is a subset of the field $\mathscr{F}$, containing 0 and closed under multiplication then the relation

$$\text{“}R(x, y) = 1 \textit{ if and only if } y - x \in \mathscr{P}\text{”}$$

gives, obviously, a reflexive relation on $\mathscr{F}^+$. Such relations will be called *reflexive relations* on the *field* $\mathscr{F}$.

Clearly, the results formulated in Propositions 4 to 7 are valid for fields, more exactly, for the additive group of fields.

We shall first consider tournaments on a field.

Theorem 8. *One can define a tournament on a field $\mathcal{F}$ if and only if $\mathcal{F}$ has no element whose square is equal to* -1.

Proof. Let R be a tournament on $\mathcal{F}$. Then, we get as a consequence of the trichotomy that either $R(0, a)=1$ or $R(0, -a)=R(a, 0)=1$ holds. For any $a\in\mathcal{F}$, $R(0, a^2)=1$ is implied in both cases. From $R(0, 1)=1$ follows, again by trichotomy, that $R(0, -1)$ is not equal to 1. This means that -1 is not the square of any element of the field.

Now, for the proof of the second part, let us suppose that $\mathcal{F}$ has the property required. Let $\mathcal{F}^\times$ denote the multiplicative group of $\mathcal{F}$ and let $\mathcal{M}_0^\times$ denote the set of squares of elements of $\mathcal{F}^\times$. By hypothesis, the subgroup $\mathcal{M}_0^\times$ of $\mathcal{F}^\times$ does not contain -1. Let $\mathcal{M}^\times$ be chosen as a maximal group extension of $\mathcal{M}_0^\times$ which does not contain -1. Then for any $a\in\mathcal{F}^\times$ from $a\notin\mathcal{M}^\times$ it follows that $-1\in\{a, \mathcal{M}^\times\}$, i.e. $-1=am$ where $m\in\mathcal{M}^\times$, since $a^2\in\mathcal{M}^\times$. So, $a=-m^{-1}$ lies in $\{-1, \mathcal{M}^\times\}$, from which, for $\mathcal{M}=\{0\}\cup\mathcal{M}^\times$, it follows:

(i) $\mathcal{M}$ is closed under multiplication (and division),

(ii) $\mathcal{M}\cap-\mathcal{M}=\{0\}$,

(iii) $\mathcal{M}\cup-\mathcal{M}=\mathcal{F}$.

Hence the relation

$$\text{“}R(a, b)=1 \quad\text{if and only if}\quad b-a\in\mathcal{M}\text{”}$$

is a tournament on $\mathcal{F}$.

One can easily verify the following:

Proposition 9. *For a tournament R on a field $\mathcal{F}$ the following assertions are equivalent:* (i) *R is a partial order,* (ii) *R is a full order,* (iii) *R is transitive,* (iv) *$\mathcal{P}_R$ is closed under addition.*

Proposition 10. *A full order on a field is also a tournament.*

These results show that tournaments can be regarded as an immediate generalization of full orders.

It is useful to give some examples.

Example 1. Let $\mathcal{Q}$ be the field of rationals. To determine a tournament R on $\mathcal{Q}$ is necessary and sufficient to decide, for each prime p, whether p or $-p$ should belong to $\mathcal{P}_R$. All combinations of such decisions define exactly one tournament on $\mathcal{Q}$.

Example 2. The field $\mathcal{F} = \mathcal{Q}(\sqrt{-2})$ is not a formal-real one; i.e. there exists no full order on $\mathcal{F}$, since $(\sqrt{-2})^2 + 1^2 = -1$. However, one can give tournaments on $\mathcal{F}$, for $i \notin \mathcal{F}$. One can construct the tournaments on $\mathcal{F}$ using example 1.

Example 3. Let $\mathcal{F}$ be a finite field with characteristic p and with order p^k. The group $\mathcal{F}^\times$ is cyclic, and thus there exists a tournament on $\mathcal{F}$ if and only if the order of $\mathcal{F}^\times$ is not divisible by four. This is the case if and only if $p \equiv 3 \pmod 4$ and k is an odd number. In this case the tournament is uniquely determined, since the negative of a non-square is a square.

. Example 4. Let $\mathcal{F}$ be a field of characteristic p. There exist tournament on $\mathcal{F}$ if and only if $p \equiv 3 \pmod 4$ and the degree of any algebraic element of $\mathcal{F}$ is an odd number. The tournament is uniquely determined if and only if $\mathcal{F}$ is an (absolute) algebraic field.

Example 5. Let $\mathcal{F}$ be any real-closed field. In this case not only the full order but also the tournament on $\mathcal{F}$ is uniquely determined.

Examples 3 and 4 show that tournaments are able, to some extent, to substitute for the full order if this latter does not exist. The last three examples suggest that a tournament on a field is uniquely determined if and only if the negatives of the non-squares are all squares. The proof is quite similar to the proof of Theorem 8.

There is, however, a further similarity between tournaments and full orders. For tournaments one can build a theory analogous to the theory of real-closed fields.

Call a field a *T*-field if one can define a tournament on it, and a *T*-closed field if it has no proper algebraic extension which is also a *T*-field.

The following assertions hold:

1) One can define a uniquely determined tournament on a *T*-closed field.

2) Each element of a *T*-closed field is either a square or a negative of a square.

3) Each polynomial of odd degree over a *T*-closed field has a root in the field.

4) One obtains an algebraically closed field from the field $\mathcal{F}$ by the proper adjunction of a root of the polynomial x^2+1 if and only if $\mathcal{F}$ is a *T*-closed field.

5) A *T*-closed field is a real-closed one if and only if it has characteristic 0. The characteristic of the other *T*-closed fields has form $4k+3$.

Now, let R be a layerwise order on the field $\mathcal{F}$ with positive cone $\mathcal{P}$. As was shown, those elements of $\mathcal{F}$ which are neither negative nor positive are just the pseudoidentities of the additive group of $\mathcal{F}$. For an element a of $\mathcal{F}$ the property $a + \mathcal{P} = \mathcal{P}$ is equivalent to either being a pseudoidentity or being 0. Thus, 0 and the pseudoidentities of the additive group $\mathcal{F}^+$, called the *small* elements of the layerwise order, form a subgroup $\mathcal{N}$ of $\mathcal{F}^+$.

Proposition 11. *In a layerwise ordered field the inverse of a small element cannot be small.*

Proof. Let us suppose that both a and a^{-1} lie in $\mathcal{N}$. Then, on the one hand, $a-a^{-1}$ belongs to $\mathcal{N}$ and, on the other hand, $a-a^{-1} = (1+a)(1-a^{-1})\in\mathcal{P}$ since, by the assumption, both $1+a$ and $1-a^{-1}$ lie in the positive cone. This is, however, a contradiction, $a-a^{-1} = 0$ not being possible.

Theorem 9. *Each layerwise order on a field has a unique extension to a full order.*

Proof. Let R be a layerwise order on the field $\mathcal{F}$ with positive cone $\mathcal{P}$ and with $\mathcal{N}$ being the set of small elements. If a is an elements of $\mathcal{N}$ then, by Proposition 11, either $a^{-1}\in\mathcal{P}$ or $-a^{-1}\in\mathcal{P}$ is fulfilled. If R has an extension to a full order then in this full order either a or $-a$ is a positive element. Thus, the uniqueness is proved. It still remains to prove that one can get a full order by extending R.

In order to verify this we shall define the positive cone $\bar{\mathcal{P}}$ of the full order as follows:

Let $a\in\bar{\mathcal{P}}$ if and only if there exists an element $p\in\mathcal{P}$, different from 0, such that ap belongs to $\mathcal{P}$.

I. $\bar{\mathcal{P}}$ is closed under addition and multiplication.

$a, b\in\mathcal{P}$ entails the existence of elements p, q of $\mathcal{P}$, different from 0, such that both pa and qb lie in $\mathcal{P}$. From $pq\in\mathcal{P}$, $pq\neq 0$, $pq(a+b) = q(pa)+(p(qb)\in\mathcal{P}$ and $(pq)(ab) = (pa)(qb)\in\mathcal{P}$ one can infer that both $a+b$ and ab belong to $\bar{\mathcal{P}}$.

II. $\mathcal{P}\subseteqq\bar{\mathcal{P}}$.

Indeed, in the definition of $\bar{\mathcal{P}}$, one can chose, for any $a\in\mathcal{P}$, the unity as p since both 1 and $1a$ lie in $\mathcal{P}$.

III. $\bar{\mathcal{P}}\cap -\bar{\mathcal{P}} = \{0\}$.

From a, $-a\in\bar{\mathcal{P}}$ it follows the existence of a $p\in\mathcal{P}$, different from 0, such that both pa and $-pa$ are elements of $\mathcal{P}$. This is possible, however, only in case $pa=0$, implying $a=0$.

IV. $\bar{\mathcal{P}}\cup -\bar{\mathcal{P}} = \mathcal{F}$.

From II it follows $\mathcal{P}\cup -\mathcal{P} \subseteqq \bar{\mathcal{P}}\cup -\bar{\mathcal{P}}$. Let a be a small element. Proposition 11 implies that either a^{-1} or $-a^{-1}$ belong to $\mathcal{P}$. Then, using $a(a^{-1}) = (-a)(-a^{-1}) = 1\in\mathcal{P}$, we obtain that $a\in\bar{\mathcal{P}}$ or $-a\in\bar{\mathcal{P}}$, respectively. $\mathcal{F} = \mathcal{P}\cup -\mathcal{P}\cup\mathcal{N}$ completes the proof.

Let $\mathcal{R}$ be a convex subring of the fully ordered field $\mathcal{F}$ containing the unity. Let $\mathcal{Q}$ be the set of the inverses of the positive elements of $\mathcal{R}$ completed by 0. Clearly, $\mathcal{Q}$ is a subset of the positive cone $\mathcal{P}$. The partial order the positive cone of which is equal to $\mathcal{Q}$ will be called a *coursening* of the given *full order*.

Theorem 10. *The layerwise orders of a field are just the coursenings of the full orders of this field.*

Proof. Let $\mathcal{Q}$ and $\mathcal{P}$ be the positive cones of a layerwise order and its extension to a full order on the field $\mathcal{F}$, respectively. Denote by $\mathcal{Q}^{\times}$ the elements of $\mathcal{Q}$ less 0 and by $\mathcal{R}$ the set $(\mathcal{Q}^{\times})^{-1} \cup \{0\} \cup -(\mathcal{Q}^{\times})^{-1}$. Let, further on, $a \leqq b$ denote the relation $b-a \in \mathcal{P}$, and let $a < b$ mean that $a \leqq b$ but $b \neq a$.

1) Suppose that $a \in (\mathcal{Q}^{\times})^{-1}$ and $0 < b < a$. These mean that $a^{-1} \in \mathcal{Q}$ and $a-b,\, b \in \mathcal{P}$. From $ab(b^{-1}-a) \in \mathcal{P}$ it follows, by an argument used in the proof of Theorem 9, that $b^{-1} \in \mathcal{P}$; i.e. $b^{-1}-a^{-1}$ either belongs to $\mathcal{Q}$ or is a small element. In the both cases, we obtain $b^{-1} = (b^{-1}-a^{-1})+a^{-1} \in \mathcal{Q}$, proving the convexity of $\mathcal{R}$.

2) $\mathcal{R}$ is clearly closed under multiplication.

3) Let a and b elements of $(\mathcal{Q}^{\times})^{-1}$. If both of them are small elements then both their sum and their difference belong to $\mathcal{R}$. Now, let us suppose that $0 < b \leqq a$ and that a is not a small element. This means that a belongs to $\mathcal{Q}^{\times}$ and both $a-b$ and $b+a$ are positive and less than $3a$. Thus $(3a)^{-1}$ is a positive element. If it were a small element then a^{-1} would also be a small element, i.e. a would not belong to $\mathcal{Q}^{\times}$, and this would be a contradiction. Using convexity, we get that $\mathcal{R}$ is closed under addition and subtraction.

Now, let a full order and a coursening of it be given with the positive cones $\mathcal{P}$ and $\mathcal{Q}$, respectively. $\mathcal{Q}$ is closed under multiplication, as this is an obvious consequence of its definition. Let $a \geqq b$ be positive elements of $\mathcal{R}$. Then, from $a+b \geqq 2b$ it follows $0 \leqq ab(a+b)^{-1} \leqq 2$; i.e. $\mathcal{Q}$ is closed under addition.

If $ab(a+b)^{-1} \in \mathcal{R}$ then one of a and b must be positive. Assume $a \geqq b$. If both of them are positive then the positivity of $\dfrac{2ab}{a+b}-b = b\left(\dfrac{a-b}{a+b}\right)$ and the convexity of $\mathcal{R}$ prove $b \in \mathcal{R}$. If b does not lie in $\mathcal{P}$ then we have $a < -b$ since $\dfrac{ab}{a+b}$ is a positive element. From this the positivity of $\dfrac{2ab}{a+b}-a = a\left(\dfrac{b-a}{b+a}\right)$ follows, implying $a \in \mathcal{R}$. These prove, however, that $\mathcal{Q}$ is the positive cone of a relation on the field, and this relation must be a layerwise order.

Theorem 11. *There is no proper overorder on a field, i.e. each overorder on a field is either a full order or a full overorder.*

Proof. One can prove in fact more. Let R be a reflexive and trichotomic relation on the field $\mathcal{F}$, i.e. $\mathcal{P}_R \cup -\mathcal{P}_R = \mathcal{F}$. If R is not a tournament on $\mathcal{F}$ then there is an element a, different from 0, for which both $a \in \mathcal{P}_R$ and $-a \in \mathcal{P}_R$ are fulfilled. This entails by virtue of the consequence of trichotomy that one of a^{-1} and $-a^{-1}$

belongs to $\mathscr{P}_R$, and that $-1 \in \mathscr{P}_R$, which implies $-\mathscr{P}_R \subseteqq \mathscr{P}_R$. This means that a reflexive and trichotomic relation on a field is either a tournament or a full overorder. Theorem 1 completes the proof.

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