On the socle of an object in categories

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To Professor L. Rédei on his 70th birthday

1. Introduction

In this paper for a class $\mathcal{M}$ of objects we shall define the $\mathcal{M}$-socle of an object and the $\mathcal{M}$-closure of a subobject, and we shall establish some connections between these notions.

To motivate the origin of these researches, let us mention their ring-theoretical background. According to the Wedderburn—Artin Structure Theorem, a semi-simple Artinian ring coincides with its socle which is defined by the sum of all its simple ideals. Moreover, if a (complete) direct sum of simple rings with unity is equipped with the Tychonoff topology, then its socle is a dense ideal. Hence it is an evident purpose to discuss those rings whose socle is a dense ideal.

Following [1], [3], [5] and [6], the definitions of socle and density as well as the results and proofs can be given in a quite general manner; we prove our theorems for objects of a category satisfying a certain system of axioms. After the preliminaries, in § 3 we shall prove that a semi-simple object whose socle is a dense subobject, is a special subdirect sum of simple objects, further any special object can be embedded as a dense subobject in a special semi-simple object a in such a way that they have the same socle, and this socle is a dense ideal of a. A ring-theoretical example will illustrate that this latter statement is sharp in the sense that the socle of a special semi-simple object is not necessarily a dense ideal (§ 4).

2. Preliminaries

Let $\mathcal{C}$ be a category. The objects and maps of $\mathcal{C}$ will be denoted by small Latin and small Greek letters, respectively. In this paper we adopt the notions and notations of [1], [3], [5] and [6], and we assume that the reader is familiar with them, in particular, with the concepts of monomorphism, epimorphism, subobject, kernel, ideal, image, etc. As it was done in [3], [5] and [6], we shall suppose that the category $\mathscr{C}$ satisfies some additional requirements. In the following we recall these axioms briefly. We suppose that

$\mathscr{C}$ possesses zero objects;

every map has a kernel;

every map has a normal image, and any subobject of the image has a complete counter image;

the image of an ideal by a normal epimorphism is always an ideal;

every family of objects has a (complete) direct sum and a free sum;

the class of all subobjects of any object is a complete lattice, and the set of all ideals of an object is a complete sublattice of this lattice.

In what follows, the normal image (c, ν) of a map $\alpha=\mu\nu\colon a\to b$ will be called briefly *the image of* α.

The conditions supposed before involve the validity of the *First Isomorphism Theorem* which states the following (cf. for instance [6] Theorem 2, 1):

Let $(k, \varkappa)\leqq(m, \mu)$ be two ideals of an object $a\in\mathscr{C}$ and let $\alpha\colon a\to b$ be a normal epimorphism with Ker $\alpha=(k, \varkappa)$. *If (m', μ') is the image of (m, μ) by α and $\gamma\colon a\to c$, $\gamma'\colon b\to c'$ are normal epimorphisms with* Ker $\gamma=(m, \mu)$ *and* Ker $\gamma'=(m', \mu')$ *respectively, then c and c' are equivalent objects, i.e. the commutative diagram*

$$\begin{array}{ccccc} k & \to & m & \to & m' \\ & & \downarrow\mu & & \downarrow\mu' \\ k & \xrightarrow{\varkappa} & a & \xrightarrow{\alpha} & b \\ & & \downarrow\gamma & & \downarrow\gamma' \\ & & c & & c' \end{array}$$

can be completed by an equivalence $\xi\colon c\to c'$.

Let $\mathscr{M}$ be an abstract property of simple objects of $\mathscr{C}$, i.e. there is chosen a class $\mathscr{M}$ of simple objects of $\mathscr{C}$ consisting of the objects having property $\mathscr{M}$ such that if a and b are equivalent objects then $a\in\mathscr{M}$ implies $b\in\mathscr{M}$. (An object a is called *simple* if its only ideals are $(0, \omega)$ and (a, ε_a)). An ideal (p, π) of an object a will be called an *$\mathscr{M}$-minimal ideal* of a, if $p\in\mathscr{M}$ holds.

Definition 1. The *$\mathscr{M}$-socle* (s_a, σ_a) of an object $a\in\mathscr{C}$ is the union of all $\mathscr{M}$-minimal ideals of a, and the zero ideal $(0, \omega)$ if a has no $\mathscr{M}$-minimal ideals.

The class $\mathscr{M}$ defines also a closure operation on the lattice of all subobjects of an object a. An ideal (m, μ) of an object $a\in\mathscr{C}$ will be called an *$\mathscr{M}$-maximal ideal,* if (m, μ) is the kernel of an epimorphism $\alpha\colon a\to b$ such that b belongs to $\mathscr{M}$. The set of all $\mathscr{M}$-maximal ideals forms the so called *structure $\mathscr{M}$-space M_a* of the object a.

Definition 2 (cf. [6]). The *$\mathscr{M}$-closure* $(\bar{l}, \bar{\lambda})$ of a subobject (l, λ) of $a\in\mathscr{C}$ is the intersection of all $\mathscr{M}$-maximal ideals (m, μ) containing (l, λ). If there does not exist such an ideal, then we put $(\bar{l}, \bar{\lambda})=(a, \varepsilon_a)$, and we say that (l, λ) is an *$\mathcal{M}$-dense* subobject of a.

It is obvious that the $\mathcal{M}$-closure operation is a closure operation*) indeed, but it need not be topological. The $\mathcal{M}$-closed ideals are just the so called $\mathcal{M}$-representable ideals (cf. [3]).

Throughout this paper we shall suppose that the class $\mathcal{M}$ is a modular class of simple objects, i.e. that

(i) if (p, π) is an $\mathcal{M}$-minimal ideal of an object a, then there is a unique $\mathcal{M}$-maximal ideal (m, μ) of a such that $(p, \pi)\cap(m, \mu)=(0, \omega)$;

(ii) if (l, λ) is an ideal of an object a and (q, ϑ) is an $\mathcal{M}$-maximal ideal of l, then $(q, \vartheta\lambda)$ is an ideal of a.

In [5] we defined the $\mathcal{M}$-radical $\mathcal{M}$-rad a of an object a as the intersection of all its $\mathcal{M}$-maximal ideals. The $\mathcal{M}$-radical means just the BROWN—MCOY radical determined by $\mathcal{M}$, since it is provided that $\mathcal{M}$ is a modular class (cf. SULIŃSKI [3]). The objects having zero $\mathcal{M}$-radicals, are called *$\mathcal{M}$-semi-simple* objects.

Proposition 1 ([5] Theorem 3, 6, c)). *If $\alpha\colon a\to b$ is a normal epimorphism such that $\mathcal{M}$-rad $a=\operatorname{Ker}\alpha$, then the object b is $\mathcal{M}$-semi-simple.*

In this note we shall use the notions of (complete) direct sum, discrete direct sum and special subdirect sum, respectively. We recall their definitions. An object $g\in\mathcal{C}$ is said to be a (complete) direct sum of the objects a_i, $i\in I$, if there are epimorphisms $\pi_i\colon g\to a_i$ such that for each object $h\in\mathcal{C}$ and for any system of maps $\alpha_i\colon h\to a_i$, $i\in I$, there is a unique map (the canonical map) $\gamma\colon h\to g$ such that $\gamma\pi_i=\alpha_i$ holds for all $i\in I$. Now any object a_i can be embedded in g as an ideal by a monomorphism ϱ_i such that $\varrho_i\pi_i=\varepsilon_{a_i}$ and $\varrho_i\pi_j=\omega$ $(i\neq j;\ i, j\in I)$. This direct sum will be denoted by $g=\prod_{i\in I} a_i(\pi_i, \varrho_i)$.

We need also

Proposition 2 ([6] Corollary to Theorem 3). *If a is a direct sum of objects belonging to $\mathcal{M}$, then any $\mathcal{M}$-closed ideal of a is a direct summand of a.*

Let (a, α) be the union of all ideals (a_i, ϱ_i) of $g=\prod_{i\in I} a_i(\pi_i, \varrho_i)$. Then the object a is called a *discrete direct product* of the objects a_i (cf. [1]).

An object b is said to be a *special subdirect sum* of objects a_i, $i\in I$, if

(1) there is a family of maps $\vartheta_i\colon a_i\to b$, $\tau_i\colon b\to a_i$, $i\in I$, such that $\vartheta_i\tau_i=\varepsilon_{a_i}$ and $\vartheta_i\tau_j=\omega$ for $i\neq j;\ i, j\in I$;

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*) In the structure $\mathcal{M}$-space M_a, too, there is defined a closure operation (cf. SULIŃSKI [3]). If $N\subseteq M_a$, then the closure $\bar{N}$ of N is the set of all $\mathcal{M}$-maximal ideals which contain the intersection of all ideals belonging to N. It is remarkable that there is a Galois connection between the closed subsets of M_a and the $\mathcal{M}$-closed ideals of a defined by the correspondence $\bar{N}\to(l,\lambda)=\bigcap_{(m,\mu)\in\bar{N}}(m, \mu)$.

(2) if $\alpha\tau_i=\beta\tau_i$ for each $i\in I$, where $\alpha: c\to b$, $\beta: c\to b$, then $\alpha=\beta$ follows.

This special subdirect sum will be denoted by $b=\sum_{i\in I} a_i(\vartheta_i, \tau_i)$. In the ring-theory the notion of special subdirect sum is due to McCoy [2], this definition was given by Tsalenko [4].

The *annihilator* (m^*, μ^*) of an $\mathcal{M}$-maximal ideal (m, μ) is the intersection $\bigcap(m_i, \mu_i)$ of all $\mathcal{M}$-maximal ideals $(m_i, \mu_i)\neq(m, \mu)$. Suliński [3] has proved

Proposition 3 ([3] Prop. 5, 4). *Let (m, μ) be an $\mathcal{M}$-maximal ideal of an $\mathcal{M}$-semi-simple object a. If the annihilator $(m^*, \mu^*)\neq(0, \omega)$, then (m^*, μ^*) is an $\mathcal{M}$-minimal ideal of a.*

Let a be an $\mathcal{M}$-semi-simple object and let D_a be the set of all $\mathcal{M}$-maximal ideals such that $(m^*, \mu^*)\neq(0, \omega)$. The object a is called *special* if the intersection of all $\mathcal{M}$-maximal ideals belonging to D_a is $(0, \omega)$. (Cf. Suliński [3]). An essential connection between special $\mathcal{M}$-semi-simple objects and special subdirect sums is established in

Proposition 4 (Suliński [3] Theorem 5,7). *An $\mathcal{M}$-semi-simple object a is special if and only if a is a special subdirect sum $\sum_{i\in I} a_i(\vartheta_i, \tau_i)$ of some objects $a_i\in\mathcal{M}$, moreover (a_i, ϑ_i), $i\in I$ are all $\mathcal{M}$-minimal ideals of a.*

The last statement turns out from the proof of Theorem 5,7 of [3].

3. Dense socles

Let $\mathcal{M}$ be a modular class of objects. The $\mathcal{M}$-socle of an object $a\in\mathscr{C}$ will be denoted by (s_a, σ_a), and its $\mathcal{M}$-closure by $(\bar{s}_a, \bar{\sigma}_a)$. This section is devoted to the investigation of objects whose $\mathcal{M}$-socle is an $\mathcal{M}$-dense ideal. First we prove

Theorem 1. *Let $a\in\mathscr{C}$ be an $\mathcal{M}$-semi-simple object. If the $\mathcal{M}$-socle of a is $\mathcal{M}$-dense in a, i.e. $(\bar{s}_a, \bar{\sigma}_a)=(a, \varepsilon_a)$, then the object a is special.*

Proof. If a has no $\mathcal{M}$-minimal ideals, then $(s_a, \sigma_a)=(0, \omega)$ and $(\bar{s}_a, \bar{\sigma}_a)=(a, \varepsilon_a)$ imply that the structure $\mathcal{M}$-space of a is the void set, and so $(a, \varepsilon_a)=\mathcal{M}\text{-rad}\, a$. Since a is also $\mathcal{M}$-semi-simple, we have $(a, \varepsilon_a)=(0, \omega)$.

If $(p, \pi)\neq(0,\omega)$ is an $\mathcal{M}$-minimal ideal of a, then by (1) there exists a unique $\mathcal{M}$-maximal ideal (m, μ) of a such that $(p, \pi)\cap(m, \mu)=(0, \omega)$. So (p, π) is contained in any other $\mathcal{M}$-maximal ideal (m_i, μ_i), $i\in I$, of a, and therefore we obtain

$$(0, \omega)\neq(p, \pi)\leqq\bigcap_{i\in I}(m_i, \mu_i)=(m^*, \mu^*),$$

where (m^*, μ^*) denotes the annihilator of (m, μ). Taking into account Proposition 3, (m^*, μ^*) is an $\mathcal{M}$-minimal ideal of a, and so it follows $(p, \pi)=(m^*, \mu^*)$.

Consider the intersection (d, δ) of all $\mathcal{M}$-maximal ideals having non-zero annihilator. By the consideration made above (d, δ) cannot contain $\mathcal{M}$-minimal ideals. Suppose $(d, \delta)\neq(0, \omega)$. Now because of the $\mathcal{M}$-semi-simplicity of a, there exists an $\mathcal{M}$-maximal ideal (m_0, μ_0) whose annihilator is zero. Therefore (m_0, μ_0) contains every $\mathcal{M}$-minimal ideal of a, so we obtain

$$(\bar{s}_a, \bar{\sigma}_a) \leqq (m_0, \mu_0) < (a, \varepsilon_a),$$

contradicting our assumption. Hence $(d, \delta)=(0, \omega)$ is valid which means that a is special.

The following generalization of Theorem 1 is also true.

Theorem 2. *If $a\in\mathcal{C}$ is an object satisfying $(\bar{s}_a, \bar{\sigma}_a)=(a, \varepsilon_a)$, and $\alpha\colon a\to b$ is a normal epimorphism with* $\operatorname{Ker}\alpha=\mathcal{M}\text{-rad}\, a$, *then b is a special $\mathcal{M}$-semi-simple object.*

Proof. At first we remark that $(\bar{s}_b, \bar{\sigma}_b)=(b, (\varepsilon_b)$ holds. Otherwise, there would be an $\mathcal{M}$-maximal ideal (m', μ') of b containing all of its $\mathcal{M}$-minimal ideals. Thus the First Isomorphism Theorem implies that the complete counterimage (m, μ^*) of (m', μ') is an $\mathcal{M}$-maximal ideal of a containing (s_a, σ_a) which is a contradiction. Since by Proposition 1 b is $\mathcal{M}$-semi-simple, the statement follows immediately from Theorem 1.

Though the converse statement of Theorem 1 is not true (see Theorem 4), we can prove an embedding theorem as follows.

Theorem 3. *Let $a\in\mathcal{C}$ be an $\mathcal{M}$-semi-simple object. If a is special, then a can be embedded by a monomorphism α in an object c such that*

1) c is a special $\mathcal{M}$-semi-simple object, moreover, it is a direct sum of objects belonging to $\mathcal{M}$;

2) the $\mathcal{M}$-socles of a and c are the same in the sense that $(s_a, \sigma_a\alpha)=(s_c, \sigma_c)$;

3) (a, α) as well as (s_c, σ_c) are $\mathcal{M}$-dense subobjects of c (i.e. $(\bar{a}, \bar{\alpha})=(\bar{s}_c, \bar{\sigma}_c)=(c, \varepsilon_c)$ holds).

Proof. Since a is special, by Proposition 4 a is a special subdirect sum $\sum_{i\in I} a_i(\vartheta_i, \tau_i)$ of objects $a_i\in\mathcal{M}$, and (a_i, ϑ_i) are all of the $\mathcal{M}$-minimal ideals of a. Thus the $\mathcal{M}$-socle (s_a, σ_a) of a is just $\bigcup_{i\in I}(a_i, \vartheta_i)$. Consider the canonical map $\alpha\colon a\to c=\prod_{i\in I} a_i(\pi_i, \varrho_i)$. If we set $(k, \varkappa)=\operatorname{Ker}\alpha$, then $\varkappa\tau_i=\varkappa\alpha\pi_i=\omega=\omega\tau_i$ is valid for all $i\in I$. So by the definition of the special subdirect sum we get $\varkappa=\omega$, hence α is a monomorphism. Moreover, c is a special $\mathcal{M}$-semi-simple object.

Now we turn to prove $(s_a, \sigma_a\alpha)=(s_c, \sigma_c)$. Obviously $(a_i, \vartheta_i\alpha)=(a_i, \varrho_i)$ is an $\mathcal{M}$-minimal ideal of c for each $i\in I$, and so $(s_a, \sigma_a\alpha)\leqq(s_c, \sigma_c)$ holds. Suppose $(s_a, \sigma_a\alpha)\neq(s_c, \sigma_c)$. Then there exists an $\mathcal{M}$-minimal ideal (p, π) of c differing from each (a_i, ϱ_i). By (i) there exists a unique $\mathcal{M}$-maximal ideal (m, μ) of c satisfying $(p, \pi)\cap(m, \mu)=(0, \omega)$. Thus (p, π) is contained in any other $\mathcal{M}$-maximal ideal of c. Since $m_i=\prod_{i\neq j\in I} a_j(\pi_j, \varrho_j)$ can be embedded in c by a monomorphism μ_i as an $\mathcal{M}$-maximal ideal, so we obtain $(p, \pi)\leqq\cap(m_i, \mu_i)=(0, \omega)$ which is a contradiction. Hence $(s_a, \sigma_a\alpha)=(s_c, \sigma_c)$ is proved.

To show 3), assume $(\bar{s}_c, \bar{\sigma}_c)<(c, \varepsilon_c)$. Now c has an $\mathcal{M}$-maximal ideal (m, μ) containing the $\mathcal{M}$-socle (s_c, σ_c) of c. According to Proposition 2, (m, μ) is a direct summand of c, and so $c=m\times p$ $(\varphi_1, \varphi_2; \mu, \pi)$ holds. Since (m, μ) is $\mathcal{M}$-maximal, $p\in\mathcal{M}$ and (p, π) is an $\mathcal{M}$-minimal ideal of c satisfying $(p, \pi)\cap(m, \mu)=(0, \omega)$ and (m, μ) does not contain all $\mathcal{M}$-minimal ideals of c. This is a contradiction, therefore $(\bar{s}_c, \bar{\sigma}_c)=(c, \varepsilon_c)$ is valid. Since $(s_c, \sigma_c)=(s_a, \sigma_a\alpha)\leqq(a, \alpha)$ and (s_c, σ_c) is $\mathcal{M}$-dense in c, so also (a, α) is an $\mathcal{M}$-dense subobject of c.

4. Special object without dense socle

Let $\mathscr{C}_R$ be the category of rings. In this section the objects (i.e. the rings) will be denoted by capital Latin letters. If $\mathcal{M}$ denotes the classe of all simple rings with unity, then $\mathcal{M}$ is a modular class of objects of $\mathscr{C}_R$, and the $\mathcal{M}$-radical becomes the well-known Brown—McCoy radical. The $\mathcal{M}$-socle of a ring means the sum of all its simple ideals with unity.

We shall show that Theorem 3 is sharp in the following sense.

Theorem 4. *In $\mathscr{C}_R$ there does exist a special $\mathcal{M}$-semi-simple ring A such that the $\mathcal{M}$-socle S of A is not $\mathcal{M}$-dense in A.*

Let F be a field (which is clearly a simple ring with unity) and form the complete direct sum $B=\prod_{i=1}^{\infty} F_i$ of infinitely many copies of F. Consider the ring A consisting of all vectors $b=(\dots, b_i, \dots)\in B$ for which $b_i=b_j$ whenever $i, j\geqq n_b$ for some natural number n_b depending on b. Clearly A contains the discrete direct sum $M=\bigcup_{i=1}^{\infty} F_i$ of infinitely many copies of F as an ideal, and so A is a special subdirect sum $A=\sum_{i=1}^{\infty} F_i$.

The factoring A/M obviously consists of the cosets $(a, \dots, a, \dots)+M$, and therefore $A/M\cong F$ is valid. Since F is a field, so M is a $\mathcal{M}$-maximal ideal of A.

Let P be an arbitrary $\mathcal{M}$-minimal ideal of A. If $0 \neq p = (\ldots, p_i, \ldots) \in P$, then at least one component p_i differs from 0. For any $b \in F$ and $b_0 = (0, \ldots, 0, bp_i^{-1}, 0, \ldots) \in A$ we have $(0, \ldots, 0, b, 0, \ldots) = b_0 p \in P$, therefore the i-th component F_i of A is contained in P, and so $P = F$ holds. Hence the $\mathcal{M}$-socle of A is just the discrete direct sum $M = \bigcup_{i=1}^{\infty} F_i$ which is not dense in A.

Thus this construction proves the statement.

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