Systems of equations over semigroups

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To Professor L. Rédei on his seventieth birthday

In her papers [3], [4], M. ERDÉLYI has studied systems of equations over non-commutative groups. Although her results were achieved in a group theoretical way, analogous statements can be proved for semigroups too. Namely we are going to characterize compatible systems of equations over semigroups and give a criterion for a semigroup to be $\mathfrak{n}$-algebraically closed. Moreover, we present simple descriptions for algebraically closed and weakly algebraically closed semigroups.

For the notions and notations of this paper not defined here, see CLIFFORD—PRESTON [2].

1. Preliminaries. Let S be an arbitrary semigroup, and $X=\{x_\gamma: \gamma\in\Gamma\}$ a set of symbols which will be fixed in the sequel. Any relation on the free product $S*\mathscr{F}_X$, where $\mathscr{F}_X$ denotes the free semigroup on X, will be called a *system of equations over* S with unknowns from X. Systems of equations will be denoted by small Greek letters or they are given in the form

$$f_\lambda(x_\gamma) = g_\lambda(x_\gamma) \quad (\lambda\in\Lambda,\ \gamma\in\Gamma). \tag{1}$$

More exactly, let (1) be the detailed description of a system ϱ of equations over S in that case if for all $\lambda\in\Lambda, f_\lambda(x_\gamma)$ and $g_\lambda(x_\gamma)$ are elements of $S*\mathscr{F}_X$, and any pair of elements (u, v) with $u, v\in S*\mathscr{F}_X$ belongs to ϱ if and only if there exists a $\lambda\in\Lambda$ such that $u=f_\lambda(x_\gamma)$ and $v=g_\lambda(x_\gamma)$.

Let T be a semigroup containing S. We call the system $\{t_\gamma: \gamma\in\Gamma\}$ a solution of (1), if for each $\lambda\in\Lambda$ we have $t_\gamma\in T$ and $f_\lambda(t_\gamma)=g_\lambda(t_\gamma)$. A system ϱ of equations over S will be said to be *compatible (strongly compatible)* if there exists a semigroup containing S (containing S as an ideal), in which ϱ has a solution.

A semigroup S will be called *algebraically closed (weakly algebraically closed)* if every compatible (strongly compatible) system of equations over S has a solution in S. Furthermore, let $\mathfrak{n}$ be an arbitrary infinite cardinal. We say that S is $\mathfrak{n}$-*algebraically* *closed* if every compatible system of equations over S consisting of fewer equations than $\mathfrak{n}$ has a solution in S.

A subsemigroup S of a semigroup T will be called a *retract* of T if there exists an idempotent endomorphism of T onto S (cf. [1]). A semigroup T is a *pure extension* of its subsemigroup S if every system of equations over S which has a solution in T has a solution in S, too. It is clear from the definitions above that a semigroup S is algebraically closed if and only if any semigroup T containing S is a pure extension of S.

Let S be a subsemigroup of T and φ a congruence on T. If for any $s_1, s_2 \in S$, $s_1 \equiv s_2(\varphi)$ implies $s_1 = s_2$ then we say that φ *separates* S.

Since a relation on a set M is a subset of $M \times M$, we may speak of the power of a relation. Let T be a semigroup and σ a congruence on T. We say that σ is an $\mathfrak{n}$-*congruence* if there exists a relation with cardinality $\mathfrak{n}$ on T which generates σ. Let now S be a subsemigroup of the semigroup T and let $t_\gamma \in T$ $(\gamma \in \Gamma)$ be elements, such that $S \cup \{t_\gamma: \gamma \in \Gamma\}$ generates T. In this case we say that $\{t_\gamma: \gamma \in \Gamma\}$ *generates T over S*. Using the notation $X = \{x_\gamma: \gamma \in \Gamma\}$, there exists a homomorphism φ of $S * \mathscr{F}_X$ onto T such that for all $s \in S$, $s\varphi = s$ and for all $\gamma \in \Gamma$, $x_\gamma \varphi = t_\gamma$ hold. If the congruence φ_1 on $S * \mathscr{F}_X$ corresponding to φ is an $\mathfrak{n}$-congruence, then we say that T is an $\mathfrak{n}$-*extension* of S. Naturally, it is possible that T is an $\mathfrak{n}$-extension and an $\mathfrak{m}$-extension of S with distinct cardinals $\mathfrak{n}$ and $\mathfrak{m}$. φ and φ_1 will be called the *canonical homomorphism* and *canonical congruence for* $\{t_\gamma: \gamma \in \Gamma\}$, respectively.

2. Results. The following theorem which characterizes the compatible systems of equations, is an analogue of some earlier results for modules, rings and groups (cf. [5], [6], [9] and the Lemma in [4]).

Theorem 1. *A system ϱ_0 of equations given by* (1) *over a semigroup S is compatible if and only if the congruence ϱ on $S * \mathscr{F}_X$ generated by ϱ_0 separates S.*

Proof. Let ϱ_0 be compatible. We may suppose without loss of generality that ϱ_0 is reflexive and symmetric. Let now $s, t \in S$ and $s \equiv t(\varrho)$. We have to prove that $s = t$. In accordance with Theorem 1. 8 in [2] there exist a natural number n and elements $s_0 = s, s_1, \ldots, s_n = t$ in $S * \mathscr{F}_X$ such that for any j $(1 \leqq j \leqq n)$ the following assertion holds: there exists a $\lambda_j \in \Lambda$ and elements h_j, $k_j \in (S * \mathscr{F}_X)$ for which the equalities $s_{j-1} = h_j \cdot f_{\lambda_j}(x_p) \cdot k_j$, $s_j = h_j \cdot g_{\lambda_j}(x_\gamma) \cdot k_j$ are fulfilled.

Let now T be a semigroup containing S in which ϱ_0 has a solution and let $\{t_\gamma: \gamma \in \Gamma\}$ be such a solution. We may assume that $\{t_\gamma: \gamma \in \Gamma\}$ generates T over S. Denote by φ the canonical homomorphism for $\{t_\gamma: \gamma \in \Gamma\}$; then for any $\lambda \in \Lambda$, by the definitions of φ and the t_γ's, $f_\lambda(x_\gamma)\varphi = f_\lambda(x_\gamma\varphi) = f_\lambda(t_\gamma) = g_\lambda(t_\gamma) = g_\lambda(x_\gamma\varphi) = g_\lambda(x_\gamma)\varphi$ holds. Hence for $j = 1, \ldots, n$, we have $s_{j-1}\varphi = h_j\varphi \cdot f_{\lambda_j}(x_\gamma)\varphi \cdot k_j\varphi = h_j\varphi \cdot g_{\lambda_j}(x_\gamma)\varphi \cdot k_j\varphi = s_j\varphi$ and thus $s = s\varphi = s_0\varphi = s_1\varphi = \ldots = s_n\varphi = t\varphi = t$.

In order to prove the sufficiency let χ denote the natural homomorphism of $S*\mathcal{F}_X$ upon $S*\mathcal{F}_X/\varrho$. Since ϱ separates S, χ is an embedding of S into $S*\mathcal{F}_X/\varrho$. On the other hand, $\{x_\gamma\chi: \gamma\in\Gamma\}$ is a solution of ϱ_0 in $S*\mathcal{F}_X/\varrho$. Indeed, for any $\lambda\in\Lambda$, we have $f_\lambda(x_\gamma\chi)=f_\lambda(x_\gamma)\chi=g_\lambda(x_\gamma)\chi=g_\lambda(x_\gamma\chi)$. Hence ϱ_0 is compatible.

Our next aim is to give a description of the algebraically closed and weakly algebraically closed semigroups. To this purpose first we prove a preparatory

Lemma. *Let S be a subsemigroup of a semigroup T. S is a retract of T if and only if T is a pure extension of S* (cf. Theorem 1 in [4]).

Proof. Let φ be an idempotent endomorphism of T onto S, and let (1) be a system of equations over S which has a solution $h_\gamma(\gamma\in\Gamma)$ in T. Then $h_\gamma\varphi(\gamma\in\Gamma)$ is also a solution of (1). Indeed, for any $\lambda\in\Lambda, f_\lambda(h_\gamma\varphi)=f_\lambda(h_\gamma)\varphi=g_\lambda(h_\gamma)\varphi=g_\lambda(h_\gamma\varphi)$ holds. Thus (1) has a solution *in S too.*

Conversely, suppose that T is a pure extension of S. Denote by $t_\lambda(\gamma\in\Gamma)$ the elements of $T\setminus S$. Then $\{t_\gamma: \gamma\in\Gamma\}$ generates T over S, and let φ be the canonical homomorphism for $\{t_\gamma: \gamma\in\Gamma\}$. The corresponding canonical congruence φ_1 is a system of equations over S, which has a solution in T, namely $\{t_\gamma: \gamma\in\Gamma\}$. Indeed, if $f_\lambda(x_\gamma)\equiv g_\lambda(x_\gamma)(\varphi_1)$ then $f_\lambda(t_\gamma)=f_\lambda(x_\gamma\varphi)=f_\lambda(x_\gamma)\varphi=g_\lambda(x_\gamma)\varphi=g_\lambda(x_\gamma\varphi)=g_\lambda(t_\gamma)$. In view of the purity of T it follows that φ_1 has a solution in S as well. Denote by $\{s_\gamma: \gamma\in\Gamma\}$ this solution. We now show that the mapping σ of T onto S defined by the equations $t_\gamma\sigma=s_\gamma$ for all $\gamma\in\Gamma$ and $s\sigma=s$ for all $s\in S$, is an idempotent endomorphism of T onto S. It is obvious that σ is an idempotent mapping of T onto S. Let us consider the elements $u, v, w\in T$ and let $u_1=u_1(x_\gamma)$, $v_1=v_1(x_\gamma)$, $w_1=w(x_\gamma)\in S*\mathcal{F}_X$ satisfying $u_1\varphi=u$, $v_1\varphi=v$, $w_1\varphi=w$. Suppose that $u\cdot v=w$. Then $(u_1 v_1)\varphi=u_1\varphi\cdot v_1\varphi=u\cdot v=w=w_1\varphi$, whence $u_1v_1\equiv w_1(\varphi_1)$ that is the equation $u_1(x_\gamma)\cdot v_1(x_\gamma)=w_1(x_\gamma)$ belongs to the system φ_1 of equations. Hence $u\sigma\cdot v\sigma=u_1(x_\gamma)\varphi\sigma\cdot v_1(x_\gamma)\varphi\sigma=u_1(s_\gamma)v_1(s_\gamma)=w_1(s_\gamma)=w_1(x_\gamma)\varphi\sigma=w\sigma=(u\cdot v)\sigma$. Thus φ is homomorphic.

We now prove that there exists only trivial algebraically closed semigroups, and give a simple characterization for weakly algebraically closed semigroups.

Theorem 2. *The unique algebraically closed semigroup is the one element semigroup* (cf. Theorem 3 in [4]).

Proof. Let S be an algebraically closed semigroup. Then S^0 is a pure extension of S. By our Lemma, S^0 admits an idempotent endomorphism φ onto S. Then 0φ is a zero element of S.

By a theorem of Bruck ([2], Theorem 8. 45.), S can be embedded into a semigroup T without proper ideals. By Lemma of this paper, T admits an idempotent endomorphism ψ onto S. Let $T_0=\{t\in T: t\psi=0\varphi\}$. For any $t_0\in T_0$ and $t\in T$, we have $(t_0t)\psi=(t_0\psi)(t\psi)=(0\varphi)(t\psi)=0\varphi$, hence $t_0t\in T_0$ and similarly $tt_0\in T$. Thus T_0 is an ideal of T, whence $T_0=T$ follows. Consequently $t\psi=0\varphi$ holds for all $t\in T$ and, since ψ is onto, S consists of the single element 0φ.

Theorem 3. *A semigroup is weakly algebraically closed if and only if it has an identity element* (cf. Theorem 5 in [3]).

Proof. Let S be weakly algebraically closed. Adjoining an identity element 1 to S, we get a semigroup T containing S as an ideal. Then T is a pure extension of S. By the Lemma, T has an idempotent endomorphism φ onto S. It is obvious that 1φ is an identity element of S.

Suppose now that S is a semigroup with identity 1. Let T be a semigroup containing S as an ideal. We have to prove that T is a pure extension of S. By the Lemma, it is sufficient to prove that S is a retract of T. Define the mapping φ of T into itself in the following manner: $t\varphi=t1$ for every $t\in T$. It is clear that φ maps T onto S, and the elements of S are invariant under φ. Furthermore, let t_1 and t_2 be any elements of T. Then $(t_1t_2)\varphi=(t_1t_2)1=t_1(t_21)=t_1(1(t_21))=(t_11)(t_21)$ $=(t_1\varphi)(t_2\varphi)$. Thus φ is an idempotent endomorphism of T onto S, that is, S is a retract of T.

M. Erdélyi proved in [3] that a group is weakly algebraically closed if and only if it is complete. A similar parallelism between rings with identity element and complete groups has been discovered by J. Szendrei [8].

Finally, we characterize $\mathfrak{n}$-algebraically closed semigroups. Note that for any infinite $\mathfrak{n}$, there exist non-trivial $\mathfrak{n}$-algebraically closed semigroups. The stronger fact that any semigroup can be embedded into an $\mathfrak{n}$-algebraically closed one can easily be proved, by the method of W. R. Scott [7].

Theorem 4. *A semigroup is $\mathfrak{n}$-algebraically closed if and only if it is a retract of any of its $\mathfrak{m}$-extensions for all $\mathfrak{m}<\mathfrak{n}$* (cf. Theorem 2 in [4]).

Proof. Suppose that S is a semigroup which satisfies the condition in the theorem. Let

$$f_\lambda(y_\gamma)=g_\lambda(y_\gamma)\quad(\lambda\in\Lambda,\ \gamma\in\Gamma) \tag{2}$$

be a compatible system of equations over S with $|\Lambda|=\mathfrak{m}<\mathfrak{n}$, and denote by Y the set $\{y_\gamma:\gamma\in\Gamma\}$. In virtue of Theorem 1 the $\mathfrak{m}$-congruence ψ on $S*\mathscr{F}_Y$ generated by the relation (2) separates S. Hence the natural homomorphism χ of $S*\mathscr{F}_Y$ upon $S*\mathscr{F}_Y/\psi$ induces an isomorphism of S into $S*\mathscr{F}_Y/\psi$. Thus, in the sequel we may use $\bar S=\{s\chi: s\in S\}$ instead of S.

We shall show that $S*\mathscr{F}_Y/\psi$ is an $\mathfrak{m}$-extension of $\bar S$. Observe that $\{y_\gamma\chi:\gamma\in\Gamma\}$ generates $S*\mathscr{F}_Y/\psi$ over $\bar S$. Consider the canonical homomorphism φ and the canonical congruence φ_1 for $\{y_\gamma\chi:\gamma\in\Gamma\}$. From $f(x_\gamma)\equiv g(x_\gamma)(\varphi_1), f(x_\gamma),\ g(x_\gamma)\in\bar S*\mathscr{F}_X$

it follows $f(y_\gamma)\chi = f(y_\gamma\chi) = f(x_\gamma)\varphi = g(x_\gamma)\varphi = g(y_\gamma\chi) = g(y_\gamma)\chi$, that is $f(y_\gamma) \equiv g(y_\gamma)(\psi)$ and vice versa. Thus φ_1 is an $\mathfrak{m}$-congruence. Therefore, $\bar{S}$ is a retract of $S * \mathscr{F}_Y/\psi$ and, by the Lemma, $S * \mathscr{F}_Y/\psi$ is a pure extension of $\bar{S}$. Since (2) obviously has a solution in $S * \mathscr{F}_Y/\psi$ namely $y_\gamma\chi(\gamma \in \Gamma)$, (2) has a solution in $\bar{S}$ too, and thus, $\bar{S}$ is $\mathfrak{n}$-algebraically closed.

On the other hand, let S be $\mathfrak{n}$-algebraically closed. We shall show that any $\mathfrak{m}$-extension T of S ($\mathfrak{m} < \mathfrak{n}$) has an idempotent endomorphism onto S. Consider a set of elements $\{t_\gamma : \gamma \in \Gamma\}$ generating T over S, such that the canonical congruence φ_1 for $\{t_\gamma : \gamma \in \Gamma\}$ is an $\mathfrak{m}$-congruence, that is, there exists a relation ϱ of cardinality $\mathfrak{m}$ on $S * \mathscr{F}_X$ generating φ_1. We may assume that ϱ has the form (1). Then for all $\lambda \in \Lambda$, the canonical homomorphism φ for $\{t_\gamma : \gamma \in \Gamma\}$ satisfies $f_\lambda(x_\gamma)\varphi = g_\lambda(x_\gamma)\varphi$, whence $f_\lambda(t_\gamma) = g_\lambda(t_\gamma)$. Thus the system ϱ consisting of $\mathfrak{m}(<\mathfrak{n})$ equations has a solution in T, namely $\{t_\gamma : \gamma \in \Gamma\}$. Since S is $\mathfrak{n}$-algebraically closed, ϱ has a solution in S as well, which will be denoted by $\{s_\gamma : \gamma \in \Gamma\}$.

Now define the correspondence σ of T into itself by the rule: for arbitrary $f(x_\gamma) \in S * \mathscr{F}_X$ let $f(t_\gamma)\sigma = f(s_\gamma)$. If $f(t_\gamma)$ is equal to $g(t_\gamma)$, then $f(x_\gamma)$ and $g(x_\gamma)$ are congruent under φ_1 and, now applying Theorem 1. 8 of [2] (as in the proof of our Theorem 1) we get $f(s_\gamma) = g(s_\gamma)$. Hence σ is a mapping which is obviously a homomorphism of T onto S. The idempotency of σ is also trivial. This completes the proof.

Theorems 1 and 4, and the Lemma may be formulated and proved for any equationally definable class of algebraic systems (instead of semigroups) in an analogous way. Theorem 2 permits no such generalization. Indeed, in the class of all abelian groups, satisfying the identity $x^p = 1$ (p prime), every group is algebraically closed.

References

[1] R. BAER, Absolute retracts in group theory, *Bull. Amer. Math. Soc.*, **52** (1946), 501—506.

[2] A. H. CLIFFORD—G. B. PRESTON, *The algebraic theory of semigroups.* I—II (Providence, 1961—67).

[3] M. ERDÉLYI, Systems of equations over noncommutative groups (Hungarian with English summary), *Acta Univ. Debrecen,* **6**/2 (1959—60), 43—54.

[4] M. ERDÉLYI, On n-algebraically closed groups, *Publ. Math. Debrecen,* **7** (1960), 310—315.

[5] A. KERTÉSZ, Systems of equations over modules, *Acta Sci. Math.,* **18** (1957), 207—234.

[6] G. POLLÁK, Lösbarkeit eines Gleichungssystems über einem Ringe, *Publ. Math. Debrecen,* **4** (1955), 87—88.

[7] W. R. SCOTT, Algebraically closed groups, *Proc. Amer. Math. Soc.,* **2** (1951), 118—121.

[8] J. SZENDREI, On rings admitting only direct extensions, *Publ. Math. Debrecen,* **3** (1953—54), 180—182.

[9] O. VILLAMAYOR, Sur les équations et les systèmes linéaires dans les anneaux associatifs, *C. R. Acad. Sci. Paris,* **240** (1955), 1681—1683.

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