The shell of a Hilbert-space operator. II

By CHANDLER DAVIS in Toronto (Canada)

As in the first article [3], the subject is relations between an operator A on Hilbert space $\mathfrak{H}$ and a certain subset of real 3-space, whose definition (1. 3) is reproduced below. The main purpose of the present article is to give new information on what subsets of 3-space can arise from the construction. The paper concludes with discussion of a natural conjecture concerning the relation between the shell and spectral sets (§12); of the classes C_ϱ of Sz.-Nagy and Foiaş, and of the relationship between the shell and the property of being similar to a contraction. I want to express my appreciation for conversations with E. DURSZT, which led to significant improvements in the paper.

References to §§1—8 are to the first article; the sections of the present paper are numbered beginning with §9.

Recall that the "shell" of an operator A is the set of all points

$$(\zeta, h) = \varphi(y, x) = \left(\frac{2x^*y}{\|x\|^2+\|y\|^2}, \frac{-\|x\|^2+\|y\|^2}{\|x\|^2+\|y\|^2}\right) \tag{1. 3}$$

for $x, y \in \mathfrak{H}$ and $y = Ax$. (Notations: x^*y is the complex inner product; so the first component ζ in (1. 3) is a complex number thought of as the first two real components of $\varphi(y, x) \in \mathbf{R}^3$.) More generally, if $\mathfrak{A}$ is a closed linear relation in $\mathfrak{H}$, its shell is the set of all points (1. 3) for $(y, x) \in \mathfrak{A} \subseteq \mathfrak{H} \oplus \mathfrak{H}$. The shell is denoted $s(A)$ or $s(\mathfrak{A})$; B denotes the unit ball in $\mathbf{R}^3$, and S its boundary.

9. Operators on 2-space

Let us first consider 2-dimensional $\mathfrak{H}$. All relations $\mathfrak{A}$ then fall into a few types, whose shells will be classified completely.

Type 1. *Normal operators.* The shell, as was already pointed out in §4, Example 1, is the convex hull of the points of S which correspond under stereographic projection (1. 1) to points of the spectrum.

Because of Thm. 5. 1, it is natural to include also under Type 1 all relations which are Möbius transforms of normal operators. This gives two types of "multi-valued operators": one relation with one-point spectrum $\{\infty\}$, namely

$$\{(y, 0) : y \in \mathfrak{H}\}; \tag{9.1}$$

and a class of relations with two-point spectra, namely

$$\{(\lambda x, x) : x \in \mathfrak{N}\} + \{(y, 0) : y \perp \mathfrak{N}\} \tag{9.2}$$

for arbitrary fixed $\lambda \in \mathbf{C}$ and 1-dimensional subspace $\mathfrak{N}$, giving spectrum $\{\lambda, \infty\}$. Thm. 5. 1 says that, for either (9. 1) or (9. 2), the shell is the convex hull of the points corresponding to the spectrum.

In the rest of the classification it is natural similarly to put together relations which are Möbius transforms of each other.

Type 2. *Operators with two-point spectrum.* By Thm. 5. 1, we can choose any two points of $\overline{\mathbf{C}}$, then pass to the general case by Möbius transformation. The simplest choice computationally — not conceptually! — is $\sigma(\mathfrak{A}) = \{0, \infty\}$. This occurs for the relation

$$\{(0, x) : x \in \mathfrak{N}\} + \{(y, 0) : y \in \mathfrak{S}\}, \tag{9.3}$$

$\mathfrak{N}$ and $\mathfrak{S}$ being fixed distinct 1-dimensional subspaces. Let θ be the number $\in [0, \pi/2]$ such that, for every $u \in \mathfrak{N}$ and $v \in \mathfrak{S}$, $|v^* u| = \cos\theta \|u\| \|v\|$. Then clearly (since $(y, x) \in \mathfrak{A}$ if and only if $x \in \mathfrak{N}$ and $y \in \mathfrak{S}$), $s(\mathfrak{A})$ is the ellipsoid of revolution $\dfrac{|\zeta|^2}{\cos^2\theta} + h^2 = 1$. (In the δ-coordinates this is $\delta_2 \delta_3 = \cos^2\theta\, \delta_1 \delta_4$.) The case $\theta = \pi/2$ is convenient to exclude, because the ellipsoid degenerates; but it is already described by (9. 2) anyway. The other extreme case, $\theta = 0$, will be treated later as Type 4.

Let us also give explicitly the shell of an operator with two-point spectrum $\{\lambda, -\lambda\}$, $0 < \lambda$. (The shell of any Type 2 operator can then be obtained from one of these by some rigid rotation of B.) By suitable choice of coordinate system in $\mathfrak{H}$, we may take the matrix of A to be

$$A = \lambda \begin{pmatrix} 0 & 1/\varkappa \\ \varkappa & 0 \end{pmatrix} \tag{9.4}$$

for some $\varkappa \geqq 1$. The shell is the ellipsoid whose upper h-intercept is the point

$$\left(0, \frac{-1 + \lambda^2 \varkappa^2}{1 + \lambda^2 \varkappa^2}\right) = \varphi\left(\begin{pmatrix} 0 \\ \lambda\varkappa \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right);$$

whose lower h-intercept is

$$\left(0, \frac{-\varkappa^2 + \lambda^2}{\varkappa^2 + \lambda^2}\right) = \varphi\left(\begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varkappa \end{pmatrix}\right);$$

and whose horizontal sections it is convenient to give parametrically as follows. If $x=\begin{pmatrix}1\\ \varkappa re^{i\delta}\end{pmatrix}$ with fixed $r>0$ and variable real δ, then $\varphi(Ax, x)$ traces an ellipse at height

$$h=\frac{-r^2\varkappa^2-1+\lambda^2(\varkappa^2+r^2)}{r^2\varkappa^2+1+\lambda^2(\varkappa^2+r^2)} \tag{9. 5}$$

and with axes terminating respectively at

$$\begin{cases}\zeta=\pm\dfrac{2\lambda r(\varkappa^2+1)}{r^2\varkappa^2+1+\lambda^2(\varkappa^2+r^2)},\\ \zeta=\pm i\dfrac{2\lambda r(\varkappa^2-1)}{r^2\varkappa^2+1+\lambda^2(\varkappa^2+r^2)}.\end{cases} \tag{9. 6}$$

In particular, of course, eigenvectors are obtained by choosing $re^{i\delta}=\pm 1$. This leads to the particular points

$$\varphi\left(\begin{pmatrix}\pm\lambda\\ \lambda\varkappa\end{pmatrix}, \begin{pmatrix}1\\ \pm\varkappa\end{pmatrix}\right)=\left(\frac{\pm 2\lambda}{1+\lambda^2}, \frac{-1+\lambda^2}{1+\lambda^2}\right)=\tau(\pm\lambda) \tag{9. 7}$$

(cf. (1. 1)), in agreement with Thm. 2. 2.

In the particular case $\lambda=1$, the ellipsoid is a prolate ellipsoid of revolution about its major axis, joining the points $(\pm 1, 0)$. This is the case where it is obtained by rigid rotation from the shell of (9. 3).

An operator on 2-space with two-point spectrum can be specified, up to isomorphisms of $\mathfrak{H}$ and Möbius transformations of the operator, by a single real parameter describing its "departure from normality". This parameter appeared as θ above and then as $\varkappa$. The relation between the two is $\cos\theta=\dfrac{\varkappa^2-1}{\varkappa^2+1}$.

Type 3. *Operators with one-point spectrum.* If $\sigma(A)=\{0\}$ then A may be represented by the matrix $\varrho\begin{pmatrix}0&0\\1&0\end{pmatrix}$. This is obtained as the limit of (9. 4) as $\varkappa\to\infty$ with $\lambda\varkappa=\varrho$ fixed. That is, we keep the ellipsoid's upper h-intercept fixed at $\left(0, \dfrac{-1+\varrho^2}{1+\varrho^2}\right)$ while its two points of tangency (9. 7) both approach $\tau(0)=(0, -1)$. It is easy to see from (9. 6) that the eccentricity of horizontal sections approaches 0. Of course it is easy to compute directly that

$$s\begin{pmatrix}0&0\\ \varrho&0\end{pmatrix}=\{(\zeta, h)\colon \varrho^2(1+\varrho^2)|\zeta|^2+((1+\varrho^2)h+1)^2=\varrho^4\}. \tag{9. 8}$$

Under Möbius transformations of the operator, all these are evidently equivalent. Any $\mathfrak{A}$ in the 2-dimensional case with one-point spectrum is equivalent to one of these under a Möbius transformation giving a rigid rotation of $\boldsymbol{B}$.

Type 4. *Operators with spectrum the extended plane.* This is an abuse of language. For any fixed 1-dimensional subspace $\mathfrak{R}$ of $\mathfrak{H}$, let $\mathfrak{A}$ be the relation $\mathfrak{R}\oplus\mathfrak{R}\subseteqq\mathfrak{H}\oplus\mathfrak{H}$. By Def. 2. 1 and Def. 2. 4, $\infty\in\sigma_p(\mathfrak{A})$ because $\mathfrak{A}$ is not an operator, i.e., $\mathfrak{N}(\mathfrak{A}^{-1})\neq\{0\}$. But also for any $z\in C$, $\mathfrak{N}(\mathfrak{A}-z\mathfrak{J})$ may be shown to be non-zero: choose $x\in\mathfrak{R}$ and it follows that $(zx, x)\in\mathfrak{A}$, $(0, x)\in\mathfrak{A}-z\mathfrak{J}$; therefore $z\in\sigma_p(\mathfrak{A})$.

In this case it is easily seen that $s(\mathfrak{A})$ is the whole unit sphere S. This is the limiting case of Type 2 in which the parameter θ occurring there approaches 0; that is, the subspaces $\mathfrak{R}$ and $\mathfrak{S}$ coalesce in (9. 3). It may also be regarded as the limiting case of Type 3 for $\varrho\to\infty$.

The above classification is complete in the following sense.

Theorem 9. 1. *Assume that* $\dim\mathfrak{H}=\dim\mathfrak{A}=2$. *If* $\sigma(\mathfrak{A})=\overline{\mathbf{C}}$ *then* $\mathfrak{A}=\mathfrak{R}\oplus\mathfrak{R}$ *for some* 1-*dimensional* $\mathfrak{R}\subseteqq\mathfrak{H}$. *Otherwise, for suitably chosen Möbius transformation* μ, *and suitably chosen coordinate system in* $\mathfrak{H}$, $\mu(\mathfrak{A})$ *is an operator with matrix*

$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \quad or \quad \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \quad or \quad \begin{pmatrix}0 & 1/\varkappa\\ \varkappa & 0\end{pmatrix} \quad (\varkappa\geqq 1).$$

Proof. Assume $\sigma(\mathfrak{A})=\overline{\mathbf{C}}$. Then both $\mathfrak{N}(\mathfrak{A})$ and $\mathfrak{N}(\mathfrak{A}^{-1})$ are non-zero, so that $\mathfrak{N}(\mathfrak{A})^{-1})\oplus\mathfrak{N}(\mathfrak{A})$ is a subspace of $\mathfrak{H}\oplus\mathfrak{H}$ having dimensionality at least 2; and it is $\subseteqq\mathfrak{A}$. By hypothesis it must be of dimensionality 2 and be $=\mathfrak{A}$. That is, $\mathfrak{A}$ is of Type 4. Now all the assertions of the theorem can be obtained at once from the foregoing analysis.

It remains to compile the facts on the more pathological cases where $\dim\mathfrak{A}\neq$ $\neq\dim\mathfrak{H}=2$. These depend upon considerations applicable regardless of the dimensionality of $\mathfrak{H}$, so I now allow $\mathfrak{H}$ to be arbitrary.

10. Geometry of the shell

If $\dim\mathfrak{A}=0$, $\mathfrak{A}=\{(0,0)\}$ and $s(\mathfrak{A})$ is empty.

If $\dim\mathfrak{A}=1$, $\mathfrak{A}$ consists of scalar multiples of a single pair (y, x), so $s(\mathfrak{A})$ is the single point $\varphi(y, x)$, which may be any point of $\boldsymbol{B}$.

Theorem 10. 1. *If* $\dim\mathfrak{A}=2$, *then* $s(\mathfrak{A})$ *is an ellipsoid (perhaps degenerate). If* $\dim\mathfrak{A}>2$, *then* $s(\mathfrak{A})$ *is convex.*

The idea with which the following proof begins is the same as that which underlies Thm. 3. 2, but the presentation seems superior to the one I used there.

Let $\mathfrak{S}$ be any subspace $\subseteqq\mathfrak{A}$, of dimensionality m, which for convenience is assumed finite. Denote by $\mathbf{C}^m$ the Hilbert space of m-component column-vectors with the usual inner product; by W, a linear isometry on $\mathbf{C}^m$ onto $\mathfrak{S}$. Since $\mathfrak{S} \subseteqq \mathfrak{H} \oplus \mathfrak{H}$, we may consider any Ww ($w \in \mathbf{C}^m$) as an ordered pair $(W_1 w, W_2 w)$; then the W_j are contractions mapping $\mathbf{C}^m$ into $\mathfrak{H}$, and $W_1^* W_1 + W_2^* W_2 = 1$.

Now an arbitrary point of B may be represented by positive-homogeneous coordinates $(\delta_1, \delta_2, \delta_3, \delta_4)$, with δ_1 and δ_4 real, with $\delta_3 = \bar{\delta}_2$, with $\delta_2 \delta_3 \leqq \delta_1 \delta_4$ and with $\delta_1 + \delta_4 > 0$; see §5 for details. In particular the representation is such that the δ-coordinates of any $\varphi(y, x)$ $(x, y \in \mathfrak{H})$ are $(y^* y, x^* y, y^* x, x^* x)$. Instead of identifying points of B with rays in a 4-dimensional cone, it is equivalent to identify them with points of a section of the cone — say, by confining attention to those $(\delta_1, \delta_2, \delta_3, \delta_4)$ for which $\delta_1 + \delta_4 = 1$. The last equation means in the case of $\varphi(y, x)$ that we deal only with pairs so $\|y\|^2 + \|x\|^2 = 1$, that is, unit vector in $\mathfrak{H} \oplus \mathfrak{H}$.

In these terms, consider an arbitrary point of $s(\mathfrak{S})$, arising from a vector-pair in the range of W. Its δ-coordinates will be of the form

$$(\delta_1, \delta_2, \delta_3, \delta_4) = (w^* W_1^* W_1 w,\ w^* W_2^* W_1 w,\ w^* W_1^* W_2 w,\ w^* W_2^* W_2 w) =$$
$$= \big(\operatorname{tr}(W_1^* W_1 w w^*),\ \operatorname{tr}(W_2^* W_1 w w^*),\ \operatorname{tr}(W_1^* W_2 w w^*),\ \operatorname{tr}(W_2^* W_2 w w^*)\big) = \Phi(w w^*),$$

where Φ is the linear map into $\mathbf{C}^4$ from the space $\mathfrak{B}(\mathbf{C}^m)$ defined by

$$\Phi(H) = \big(\operatorname{tr}(W_1^* W_1 H),\ \operatorname{tr}(W_2^* W_1 H),\ \operatorname{tr}(W_1^* W_2 H),\ \operatorname{tr}(W_2^* W_2 H)\big).$$

Actually we need consider only hermitian H; they comprise a real subspace of $\mathfrak{B}(\mathbf{C}^m)$, on which Φ is a real-linear operator; and Φ automatically maps to points $(\delta_1, \delta_2, \delta_3, \delta_4)$ satisfying $\delta_1 = \bar{\delta}_1$, $\delta_4 = \bar{\delta}_4$, $\delta_3 = \bar{\delta}_2$. If $H \geqq 0$ then from the Schwarz inequality $\delta_2 \delta_3 \leqq \delta_1 \delta_4$. Our H are still more special, being of the form ww^*, $w \in \mathbf{C}^m$. The condition that $\delta_1 + \delta_4 = 1$ as desired, is that

$$1 = w^* W_1^* W_1 w + w^* W_2^* W_2 w = w^* w.$$

Conclusion: $s(\mathfrak{S})$ is the image under a real-linear map of the set of projectors of rank 1 in $\mathfrak{B}(\mathbf{C}^m)$.

Any two non-collinear points of $\mathfrak{A}$ belong to a 2-dimensional subspace $\mathfrak{S} \subseteqq \mathfrak{A}$. We can go through the above construction with $m = 2$. Hence, any two points of $s(\mathfrak{A})$ belong to a subset of $s(\mathfrak{A})$ which is the image of an ellipsoid by a real-linear map (this is Thm. 3. 2); namely, $s(\mathfrak{S})$ is the image under Φ of $\{ww^* \colon w \in \mathbf{C}^2, \|w\| = 1\}$, which in matrix form is the set of all

$$\text{(10. 1)} \qquad \begin{pmatrix} \cos\theta \\ e^{-i\omega} \sin\theta \end{pmatrix} (\cos\theta \quad e^{i\omega} \sin\theta) = \frac{1}{2} + \frac{1}{2} \begin{pmatrix} \cos 2\theta & e^{i\omega} \sin 2\theta \\ e^{-i\omega} \sin 2\theta & -\cos 2\theta \end{pmatrix},$$

and the last matrix clearly gives, as θ and ω vary, a sphere of radius $\sqrt{2}$ in the Frobenius norm.

If $\dim \mathfrak{A} = 2$, then take $\mathfrak{S} = \mathfrak{A}$, and that finishes the first part of the proof.

If $\dim \mathfrak{A} > 2$, we will be finished once we have seen that, for every 2-dimensional $\mathfrak{S} \subseteqq \mathfrak{A}$ and corresponding ellipsoid $s(\mathfrak{S}) \subseteqq s(\mathfrak{A})$, the inner domain of $s(\mathfrak{S})$ (if any) also belongs to $s(\mathfrak{A})$. Choose any 3-dimensional subspace $\mathfrak{R}$ so $\mathfrak{S} \subseteqq \mathfrak{R} \subseteqq \mathfrak{A}$, and extend the isometry $W: \mathbf{C}^2 \to \mathfrak{S}$ to an isometry $\mathbf{C}^3 \to \mathfrak{R}$, still denoted by W. The only change entailed in the above is that (10. 1) is bordered by a third row and third column consisting of zeroes. Now the set $\{ww^*: w \in \mathbf{C}^3, \|w\| = 1\}$ contains the set of all matrices

$$(10.2) \qquad \begin{pmatrix} \cos\theta \sin\varphi \\ e^{-i\omega} \sin\theta \sin\varphi \\ \cos\varphi \end{pmatrix} (\cos\theta \sin\varphi \quad e^{i\omega} \sin\theta \sin\varphi \quad \cos\varphi) \qquad (0 \leqq \varphi \leqq \pi/2),$$

which is a 3-cell containing the set (10. 1). Therefore $s(\mathfrak{A})$, which is the image of (10. 1) under Φ, is not essentially imbedded in the image of (10. 2) under Φ, hence not essentially imbedded in $s(\mathfrak{A})$. This is enough to complete the proof.

We can now also complete the classification of shells in the special case $\dim \mathfrak{H} = 2$.

Corollary. *If* $\dim \mathfrak{A} > \dim \mathfrak{H} = 2$, *then* $s(\mathfrak{A})$ *is all of* B.

Proof. Since we now know that $s(\mathfrak{A})$ is convex, it is enough to prove that it contains all of S. Now the two subspaces $\mathfrak{A}$ and $\{0\} \oplus \mathfrak{H}$ of the 4-dimensional space $\mathfrak{H} \oplus \mathfrak{H}$ have dimensionalities totalling > 4, hence they have non-trivial intersection; that is, $\mathfrak{N}(\mathfrak{A})$ is non-zero; that is, $(0, -1) \in s(\mathfrak{A})$. For any Möbius transformation μ, $\mu(\mathfrak{A})$ also has dimensionality $\dim \mathfrak{A}$ and the same argument applies: $(0, -1) \in s(\mu(\mathfrak{A}))$. But the Möbius transforms of $(0, -1)$ comprise all of S, and the proof is complete.

As in the case of numerical range, there is also a rather trivial converse.

Theorem 10. 2. *Any convex subset of* B *is the shell of some relation.*

It is convenient to use the following fact, a sort of generalization of § 4, Example 1.

Lemma 10. 1. *Let* $\mathfrak{H}$ *be the orthogonal direct sum* $\oplus_p \mathfrak{H}_p$, *where each* $\mathfrak{H}_p$ *is a Hilbert space and* p *ranges over some index set* K. *Let the relation* $\mathfrak{A}$ *on* $\mathfrak{H}$ *be the direct sum of relations* $\mathfrak{A}_p$ *on* $\mathfrak{H}_p$. *Then* $s(\mathfrak{A})$ *is the convex hull of the* $s(\mathfrak{A}_p)$.

Proof. An arbitrary point (y, x) of $\mathfrak{A}$ is obtained by setting $x = \Sigma_p \xi_p x_p$, $y = \Sigma_p \xi_p y_p$; here for each p, (y_p, x_p) is an arbitrary non-zero point of $\mathfrak{A}_p$ but may be subjected to $\|x_p\|^2 + \|y_p\|^2 = 1$ without loss of generality, and the coefficients are arbitrary subject to $\Sigma |\xi_p|^2 < \infty$ but may be subjected to $\Sigma |\xi_p|^2 = 1$ without loss of generality. Now $x^* y = \Sigma_p |\xi_p|^2 x_p^* y_p$, a generalized convex combination; and similarly for $x^* x$ and $y^* y$ with the same coefficients. For the images under the mapping (1. 3) one computes that $\varphi(y, x)=\Sigma_p|\xi_p|^2\varphi(y_p, x_p)$. This is obviously in the closure of the convex hull of the $s(\mathfrak{A}_p)$; I omit the rather standard geometric argument which shows it is in the convex hull itself.

Proof of Thm. 10. 2. Let K be a convex subset of B. For each point p of K, let $\mathfrak{H}_p$ be a 2-dimensional Hilbert space, and let $\mathfrak{A}_p$ be a relation on $\mathfrak{H}_p$ spanned by the single pair (y_p, x_p), where x_p and y_p are so chosen in $\mathfrak{H}_p$ that $\varphi(y_p, x_p)=p$ and $\|x_p\|^2+\|y_p\|^2=1$. Application of the Lemma completes the proof.

This construction is admittedly artificial. It can be modified in several respects, but an essential feature is the large dimensionality of the co-domain $\mathfrak{H}\ominus\mathfrak{D}(\mathfrak{A})$. In the case of an everywhere-defined bounded operator, the shell is significantly less arbitrary; see Thm. 3. 3 and § 11. It would be interesting to find a satisfactory necessary and sufficient condition for a subset of B to be shell of some everywhere-defined bounded operator.

11. Cones containing the shell

Throughout this section, we consider everywhere-defined bounded operators.

Although the shell (for instance, that of a normal operator) may have sharp edges or vertices, these can occur only on S; in the interior of B, the shape of the boundary of the shell must be snub.

How can we express this geometric idea precisely? A first, qualitative expression is suggested by the known fact [9, Satz 1(i)] that every corner of the boundary of the numerical range belongs to the approximate point spectrum. That fact can be slightly strengthened, becoming the following property of the shell.

Theorem 11. 1. *Let p be a point of $B\setminus S$ which belongs to the boundary $\partial(s(A))$. Assume there is no segment having both endpoints in S and lying entirely in $\overline{s(A)}$. Then $\overline{s(A)}$ has a unique supporting plane at p.*

We know the case of whole segments as edges of the boundary does occur (§ 4), so it was necessary to exclude it.

Proof. Suppose there are distinct supporting planes at p, and let l denote their line of intersection. The hypotheses imply that at least one point of intersection of l with S is not in $\overline{s(A)}$. Since everything so far is invariant under the transformations considered in § 5, we are free to assume without loss of generality that this point is the north pole (0, 1) and that l is the vertical axis $\zeta=0$.

Then A is a bounded operator such that some $(0, h)$ $(h>-1)$ is in $s(A)$; say, $(0, h)=\varphi(y, x)$, with $y=Ax$, $x\neq 0$. Then by definition (1. 3), $h>-1$ means $y\neq 0$, and $\zeta=0$ means $y\perp x$. This must be shown to contradict the hypothesis that $s(A)$ has two vertical supporting planes through l (that is, by Thm. 3. 1, that $\partial w(A)$ has a corner at 0). For this, it is enough to show that the set of numbers $(x+te^{i\delta}y)^* \cdot A(x+te^{i\delta}y)$ (for $t>0$ and real δ) is not contained in any proper sector in $\mathbf{C}$ with opening $<\pi$.

Now by the choice of x and y,

$$(x+te^{i\delta}y)^*A(x+te^{i\delta}y) = te^{i\delta}x^*Ay+te^{-i\delta}y^*y+t^2y^*Ay, \tag{11.1}$$

where the second term is non-zero. Let δ_0 be such a value of δ as to make the coefficient of t non-zero. (Such a value exists; indeed, if perchance $e^{i\delta_1}x^*Ay+e^{-i\delta_1}y^*y=0$ then we can take $\delta_0 = \delta_1+\pi/2$ and get $|e^{i\delta_0}x^*Ay+e^{-i\delta_0}y^*y| = 2y^*y\neq 0$.) Then as $t\to 0^+$, the points (11.1) for $\delta=\delta_0$ and $\delta = \delta_0+\pi$ approach 0 along a line or parabola, from opposite sides. This completes the proof.

I now turn to the more delicate quantitative version. Whereas Thm. 11.1 says $s(A)$ cannot have a sharp point at $p\in B\setminus S$, the following theorems say that even a slightly rounded point is impossible, and give exact information on how blunt it must be. We start with a supporting plane at p; since there is no loss of generality in taking this plane horizontal, the hypothesis begins by fixing the norm of A.

Theorem 11.2. *Assume* $\|A\|=\varrho$. *If, for some real constants* r *and* k *such that* $r+k\leqq 0<r-k$, $s(A)$ *is contained in the cone* $|\zeta|\leqq r+kh$, *then* $s\begin{pmatrix}0 & 0\\ \sigma & 0\end{pmatrix}$ *is also contained in that same cone.*

In other words, the special operator $\begin{pmatrix}0 & 0\\ \sigma & 0\end{pmatrix}$ satisfies all inequalities of the form

$$\lambda|x^*Ax| \leqq \|x\|^2+\varkappa\|Ax\|^2 \qquad (\lambda>0,\ \varkappa\leqq 0) \tag{11.2}$$

which can be satisfied by any operator of the same norm.

This theorem has an equivalent, more explicit formulation, which follows. For simplicity, I state only the case $\varrho=1$.

Theorem 11.3. *Assume* $\|A\|=1$. *Let* $\pi/2>\vartheta\geqq\arcsin(1/3)$. *Then for every* $\varepsilon>0$, *there exists* $(\zeta, h)\in s(A)$ *for which* $\sqrt{2}\cos\vartheta|\zeta|+\sin\vartheta\,(2h+1)\geqq 1-\varepsilon$. *Equivalently, for every* $\varepsilon>0$ *there exists non-zero* $x\in\mathfrak{H}$ *for which*

$$(1+\varepsilon)2\sqrt{2}\cos\vartheta\,|x^*Ax| \geqq (1+\sin\vartheta)\|x\|^2+(1-3\sin\vartheta)\|Ax\|^2. \tag{11.3}$$

In this formulation, it is easy to see that the case $\vartheta=\arcsin(1/3)$ is just the known fact [8, p. 33] that the numerical radius of an operator of norm 1 is $\geqq\frac{1}{2}$. The limiting case $\vartheta\to\pi/2$ is tautological. The other cases of the theorem are believed to be new.

The equivalence of the various formulations is fairly easy to verify. The result for $\|A\|=1$ implies the generalization for arbitrary $\|A\|$ by the transformation procedure of § 5. Also, for the particular operator $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we can get equality in (11. 3) with $\varepsilon=0$, but the x which achieves this is, up to a scalar multiple, unique; from this one proves the equivalence of Thm. 11. 2 with Thm. 11. 3. Leaving to the reader the mechanical filling in of these details, I turn to the essential part, the proof of Thm. 11. 3.

By the same sort of approximation procedure as in Thm. 3. 3, we see that it is enough to prove the conclusion with $\varepsilon=0$ and to do so under the assumption that $s(A)$ is closed. In particular, we may assume that $\|A\|$ is attained.

The shell of $-A$ is obtained from the shell of A upon rotation about the h-axis through a half-turn. By Lemma 10. 1, the shell of $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ is the convex hull of the shells of A and $-A$. Accordingly, it lies in every cone symmetric about the h-axis that $s(A)$ does. That means that we can assume without loss of generality that $(0, 0)\in s(A)$, that is, that there exists a vector x_0 such that $\|x_0\|=\|Ax_0\|=1$ and $x_0^*Ax_0=0$.

Let us look at the space as the direct sum of the 1-dimensional subspaces $x_0x_0^*\mathfrak{H}$ and $Ax_0(Ax_0)^*\mathfrak{H}$ with an orthogonal complement. Thus we write

$$(11.4) \qquad x_0=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A=\begin{pmatrix} 0 & \alpha & v^* \\ 1 & 0 & 0 \\ 0 & u & B \end{pmatrix},$$

where α is a number, u and v are vectors in the complementary subspace, and B is a contraction operating on that subspace. (In writing the expression for A, I have been justified in setting some entries equal to zero by the hypothesis $\|A\|=1$.)

The procedure will be as follows. Assuming the reversed inequality

$$(11.5) \qquad 2\sqrt{2}\cos\vartheta\,|x^*Ax| \leqq (1+\sin\vartheta)\|x\|^2+(1-3\sin\vartheta)\|Ax\|^2$$

for all $x\in\mathfrak{H}$, it will be deduced that equality is attained, which constitutes proving the Theorem. It will also be deduced that A has a certain special form — that is, the extremal case in the Theorem will be identified.

As it turns out, it suffices to consider vectors of the form $x=\begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix}$. If we substitute this into (11. 5), we find on the left side a positive multiple of $|\alpha\bar{\xi}\eta+\xi\bar{\eta}|$, while on the right side all terms depend only upon the moduli of ξ and η. This allows us to replace $|\alpha\bar{\xi}\eta+\xi\bar{\eta}|$ by $-(|\alpha|+1)\,\mathrm{Re}\,(\bar{\xi}\eta)$, for both expressions assume the same maximum (namely, $(|\alpha|+1)|\xi\eta|$) when $\arg\xi$ and $\arg\eta$ are varied. The transformed inequality may be written

$$(11.6)\qquad 0 \leqq (\bar{\xi}\bar{\eta}) \begin{pmatrix} 2-2\sin\vartheta & \sqrt{2}\cos\vartheta\,(|\alpha|+1) \\ \sqrt{2}\cos\vartheta\,(|\alpha|+1) & 1+\sin\vartheta+(1-3\sin\vartheta)(|\alpha|^2+\|u\|^2) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

(assumed for all ξ, η). Remember that $1-3\sin\vartheta \leqq 0$; therefore if the hermitian form in (11. 6) is positive, so is the form

$$(11.7)\qquad \begin{pmatrix} 2-2\sin\vartheta & \sqrt{2}\cos\vartheta\,(|\alpha|+1) \\ \sqrt{2}\cos\vartheta\,(|\alpha|+1) & 1+\sin\vartheta \end{pmatrix}.$$

This has positive diagonal entries, to be sure, but its determinant is $2\cos^2\vartheta\cdot$ $\cdot(1-(|\alpha|+1)^2)$, which is never positive, and is zero if and only if $\alpha=0$.

Summing up, our assumption has implied that the form (11. 6) is degenerate, as desired; that $\alpha=0$; and that (11. 7) must actually be the same as the form in (11. 6). Except in the limiting case $\sin\vartheta=1/3$ (to which I will return presently), this implies that $u=0$.

Now we can infer that $v=0$ in (11. 4) without any further computation. Indeed, replace A, x_0, Ax_0, u respectively by A^*, Ax_0, x_0, v; all the hypotheses are unaffected, so the argument that $u=0$ yields also that $v=0$.

If $\sin\vartheta=1/3$ then (11. 6) says $2\,|x^*Ax| \leqq \|x\|^2$. In this case we do have to resort to vectors $x=\begin{pmatrix}\xi\\ \eta\\ w\end{pmatrix}$. If in particular $|\xi|=|\eta|=1$, then $\|x\|^2=2+\|w\|^2$. Using (11. 4) and the already-proved fact that $\alpha=0$, we find

$$x^*Ax=\bar{\xi}v^*w+\xi\bar{\eta}+\eta w^*u+w^*Bw.$$

By proper choice of the arguments of ξ and η, we can ensure that the first three terms on the right have the same argument; then

$$|v^*w|+1+|w^*u|-|w^*Bw| \leqq |x^*Ax| \leqq \frac{1}{2}\|x\|^2 = 1+\frac{1}{2}\|w\|^2.$$

This is unaffected if w is replaced by tw, for scalar t approaching 0. Therefore $v^*w=$ $=w^*u=0$, for every w; therefore $u=v=0$.

This completes the proof of the Theorem together with the following "uniqueness" assertion.

Theorem 11. 4. *Let $\pi/2>\vartheta\geqq\arcsin(1/3)$. Assume the norm of A is 1 and is attained. If (11. 5) holds for all x, then A has $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ as an orthogonal direct summand.*

The case $\vartheta = \arcsin(1/3)$ of this Theorem is due to J. P. WILLIAMS and T. CRIMMINS [15].

It seems reasonable to hope that still more detailed information in the same direction could be found. For example

Conjecture 11. 1 *The conclusion of Thm.* 11. 2 *holds for* $\pi/2 > \vartheta \geqq \arcsin(-1/3)$.

(It fails outside that range because of the unitary operators.)

Conjecture 11. 2. *Assume* $\|A\| = \varrho$. *Assume the maximum* h *for* $(\zeta, h) \in \overline{s(A)}$ *is attained at a unique point* $\in B \setminus S$. *Then the boundary of* $\overline{s(A)}$ *there has one principal curvature at least as small as that of* $s\begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}$ *at its topmost point.*

(The other principal curvature may be greater, see § 9.)

12. Spectral sets other than discs

Recall that Thm. 7. 2 asserts that for a disc X (that is, for a subset X of $\bar{\mathbf{C}}$ which is obtained from $D = \{z : |z| \leqq 1\}$ by Möbius transformation) and any relation $\mathfrak{A}$ we have the following equivalence: X is a spectral set (s. s.) for $\mathfrak{A}$ if and only if the shell $s(\mathfrak{A})$ is contained in the convex hull of the stereographic projection $\tau(X)$.

What if X is simply some closed proper subset of $\bar{\mathbf{C}}$? For X to be s.s. for $\mathfrak{A}$, it is still necessary that $s(\mathfrak{A})$ be contained in the convex hull of $\tau(X)$; this corresponds to the easy half of Thm. 7. 2., and the same proof applies. But is it sufficient?

In this section I give a counterexample.

As the set X, I will take the set of those z in the unit disc D such that either $|z| \leqq 1/5$ or $\arg z$ is an integral multiple of $\pi/10$: a hub with 20 spokes. (If you prefer, the spokes could be given a non-zero thickness so that X would be the closure of its interior.)

As the operator A, take $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then X is not s.s. for A, by a remarkable result of C. BERGER [1]: every simply-connected spectral set of this operator contains the disk $\{z : |z| \leqq 1/4\}$.

To show that $s(A)$ is contained in the convex hull $\operatorname{conv}(\tau(X))$, it is easiest to work with the hub and the spokes separately. First, $\tau(X) \supseteqq \tau(\{z : |z| \leqq 1/5\}) = \{(\zeta, h) \in S : h \leqq -12/13\}$, so $\operatorname{conv}(\tau(X))$ contains all points of B having $h \leqq -12/13$. It remains to consider $-12/13 \leqq h \leqq 0$. For each fixed h, in this range, the section of B is $\{\zeta : |\zeta|^2 \leqq 1 - h^2\}$ and the section of $\tau(X)$ is a set of 20 points evenly spaced around the circumference. Therefore, the section of $\operatorname{conv}(\tau(X))$

includes all points with $|\zeta|^2 \leqq \left(\cos^2 \frac{\pi}{20}\right)(1-h^2)$. The points of $s(A)$ are, by § 4 or (9. 8), those with $|\zeta|^2 \leqq -2h-2h^2$. Accordingly, it must be shown that $\cos^2 \frac{\pi}{20} \geqq \geqq \frac{-2h}{1-k}$. For the interval of h in question, the right-hand member is $\leqq \frac{24}{25}$; on the other hand, $\cos^2 \frac{\pi}{20} \geqq 1-\left(\frac{\pi}{20}\right)^2 > 1-\frac{1}{25}$, which completes the verification.

13. The classes C_ϱ

Sz.-Nagy and Foiaş [12], [13], [14], § I. 11] introduced certain classes of bounded, everywhere-defined operators A, called C_ϱ ($\varrho>0$). We say $A \in C_\varrho$ if there exists a unitary U on a Hilbert space $\mathfrak{K} \supseteqq \mathfrak{H}$ such that $A^n = \varrho \cdot pr\ U^n$ ($n=1, 2, \ldots$). They proved this is equivalent to a restriction on the values of x^*Ax and $\|Ax\|$. The latter condition was reformulated in [3, § 8] in terms of the shell, and the results may be summarized as follows:

$A \in C_\varrho$ ($\varrho \leqq 2$) if and only if for all $(\zeta, h) \in s(A)$

$$|\varrho-1| \cdot |\zeta| \leqq \varrho-1-h. \tag{13. 1}$$

$A \in C_\varrho$ ($\varrho>2$) if and only if all $(\zeta, h) \in s(A)$ satisfy (13. 1) *and, in addition,*

$$h \leqq \left(1-\frac{|\zeta|^2}{\zeta_0^2}\right)^{1/2} \quad \textit{when} \quad |\zeta| \leqq \zeta_0^2; \tag{13. 2}$$

here ζ_0^2 is the constant $\frac{\varrho(\varrho-2)}{(\varrho-1)^2}$.

The cone to which a point is restricted by (13. 1) was named K_ϱ; the cone-with-ellipsoid-cap to which a point is restricted (for $\varrho>2$) by (13. 1) and (13. 2) together was named E_ϱ.

It may be pointed out that a number of results of Durszt [4], Berger and Stampfli [2], and Furuta [5], [6] are contained in the above, or follow from it by application of general properties of the shell.

For example, they proved that, for normaloid A, $A \in C_\varrho$ if and only if

$$\|A\| \leqq \begin{cases} \frac{\varrho}{2-\varrho} & (\varrho \leqq 1) \\ 1 & (\varrho \geqq 1), \end{cases} \tag{13. 3}$$

and that (13. 3) is sufficient for $A \in C_\varrho$ even if A is not normaloid. Now in terms of the shell, a norm restriction says how high the shell may go: $\|A\| \leqq a$ restricts

$s(A)$ to $\left\{(\zeta, h): h \leqq \frac{-1+a^2}{1+a^2}\right\}$, as we see directly from (1. 3). A restriction on the spectral radius says how high the part of the shell which approaches the boundary S may go: indeed, by Thm. 2. 3, the hypothesis $\sigma_\pi(A) \subseteqq \{z: |z| \leqq a\}$ keeps $\overline{s(A)}$ from intersecting S any higher up than $\tau(a)$, that is, it restricts $\overline{s(A)} \cap S$ to lie in $\left\{(\zeta, h): h \leqq \frac{-1+a^2}{1+a^2}\right\}$. To say A is normaloid (has norm no greater than its spectral radius) is to say that at no point of $s(A)$ does h get any higher than its maximum on $\overline{s(A)} \cap S$. Accordingly, the above result of Durszt, Berger, Stampfli, and Furuta comes right out of the shell characterization of C_ϱ, once we observe that the values of h corresponding to the norm bound in (13. 3) are exactly the h-coordinates of the intersection of K_ϱ's boundary with S.

For a second example, Durszt's result that $\begin{pmatrix} 0 & 0 \\ \varrho & 0 \end{pmatrix} \in C_{\varrho'}$, if and only if $\varrho \leqq \varrho'$, reduces to the straightforward observation that the ellipse (9. 8) is tangent to the boundary of K_ϱ (and if $\varrho > 2$, the circle of tangency lies below the circle $|\zeta| = \zeta_0^2$, $h = (1-\zeta_0^2)$; see (13. 2)).

Finally, it is easy in terms of the shell to answer the natural question, what operators belong to the union of the classes C_ϱ. Durszt suggests the notation $C_\infty = \bigcup C_\varrho$.

Theorem 13.1. *The following conditions are equivalent:*

(i) $A \in C_\infty$;

(ii) $\sigma_\pi(A) \subseteqq D$, *the closed unit disc; and there does not exist any curve lying in* $s(A)$ *which approaches* S *at a point on its equator tangentially from above;*

(iii) $\sigma(A) \subseteqq D$; *and for each* θ, *the resolvent* $R_{re^{i\theta}}(A) \equiv (re^{i\theta} - A)^{-1}$ *satisfies* $\|R_{re^{i\theta}}(A)\| \leqq (r-1)^{-1}$ *for* r *in a sufficiently small interval* $]1, 1+\varepsilon[$.

By a curve approaching $(\zeta_0, 0)$ $(|\zeta_0| = 1)$ "tangentially from above" is meant here a curve $(\zeta(h), h)$ for h in some interval $]0, \varepsilon[$, such that $1 > \mathrm{Re}\,(\zeta(h)/\zeta_0) > 1 - o(h)$.

Condition (iii) has been included because it is stated without reference to the shell. Condition (ii) is essentially all in terms of the shell, because $\sigma_\pi(A) \subseteqq D$ is, as remarked before, equivalent to the condition that $h \leqq 0$ for all $(\zeta, h) \in s(A) \cap S$.

Proof of Thm. 13. 1. We know from § 8 that the equivalent conditions $\sigma_\pi(A) \subseteqq D$ and $\sigma(A) \subseteqq D$ are necessary for $A \in C_\varrho$. If $A \in C_\varrho$, then $s(A)$ lies in the cone K_ϱ and so has no curve approaching an equatorial point tangentially from above. This proves (i)$\Rightarrow$(ii).

Assume (ii); I will prove that for sufficiently large ϱ, $\overline{s(A)} \subseteqq E_\varrho$. Since $\overline{s(A)}$ is compact, it is enough to prove that, for every $p \in \overline{s(A)}$, there exists a ϱ such that E_ϱ contains some neighborhood of p in $\overline{s(A)}$. This is surely so for points with $h < 0$, for they are even in K_1. Also any $p\in B\setminus S$ gives no trouble, because it is in int E_ϱ for some ϱ. The only problem is $p=(\zeta_0, 0)$, $|\zeta_0|=1$. By (ii), we can choose a $\varrho>1$ so large that for all $(\zeta, h)\in\overline{s(A)}$, we have $(\varrho-1)\operatorname{Re}(\zeta/\zeta_0)<\varrho-1-h$. By continuity, $(\varrho-1)\operatorname{Re}(\zeta e^{i\theta}/\zeta_0)<\varrho-1-h$ for θ in some neighborhood of 0 and for all $(\zeta, h)\in\overline{s(A)}$. But (see § 8) this means that in some neighborhood of p, all points of $\overline{s(A)}$ lie in E_ϱ. Therefore (ii)$\Rightarrow$(i).

The reason (ii) and (iii) are equivalent is that the corresponding statements about a single equatorial point are equivalent. By symmetry, it does not matter which equatorial point we consider, so let us treat $(1, 0)$ and take the θ in (iii) to be 0.

Now for $r>1$,

$$\|R_r(A)\|^{-2}=\inf\left\{\frac{\|rx-Ax\|^2}{\|x\|^2}:\ x\in\mathfrak{H}\setminus\{0\}\right\}=\inf\left\{r^2+\frac{1+h-2r\operatorname{Re}\zeta}{1-h}:\ (\zeta, h)\in s(A)\right\},$$

using (1. 3). Thus (iii) will fail (for $\theta=0$) if and only if, for r arbitrarily close to 1, we can find $(\zeta, h)\in s(A)$ such that

$$\frac{1+h-2r\operatorname{Re}\zeta}{1-h}<-2r+1;$$

setting $r=1+t\ (t>0)$, this inequality reduces to $1-\operatorname{Re}\zeta<\dfrac{2t}{1+t}h$. To find (ζ, h) satisfying this for arbitrarily small $t>0$ is exactly to find (ζ, h) with $\dfrac{1-\operatorname{Re}\zeta}{h}$ arbitrarily close to 0, because all our points have $1-\operatorname{Re}\zeta>0$. If this can be done at all, it must be done for $h\to 0$, because all our points are in B; and the points can be joined by a curve because $s(A)$ is convex (or an ellipsoid). This completes the proof of the Theorem.

Corollary. *For $A\in C_\infty$, if $Ax=\lambda x$ $(|\lambda|=1)$, then $A^*x=\bar\lambda x$.*

Proof. Otherwise, $A^*x=\bar\lambda x+y$, with $y\perp x$, $y\neq 0$. Then one easily computes that for suitable $\varepsilon\to 0$, $\varphi(A(x+\varepsilon y), x+\varepsilon y)$ gives a curve violating the restriction in (ii) of the Theorem.

14. Similarity

As the property $A\in C_\varrho$ is equivalent to $s(A)$ lying in a certain subset of B, we may ask for a condition on $s(A)$ which is equivalent to A having the important related property of being similar to a contraction. The best answer available is Thm. 14. 1, which expresses in terms of $s(A)$ a condition known to be weaker. Thm. 14. 2, which follows, is in some sense complementary.

Definition 14. 1. For $\varkappa \geqq 1$, let $L_\varkappa$ denote the set of $(\zeta, h) \in B$ such that $h \leqq \dfrac{-1+\varkappa^2}{1+\varkappa^2}(1-|\zeta|^2)^{\frac{1}{2}}$.

Definition 14. 2. For $|\alpha|<1$, let μ_α denote the Möbius transformation $\mu_\alpha(z) = \dfrac{z-\alpha}{-\bar{\alpha}z+1}$.

Theorem 14. 1. *Consider the following conditions upon an everywhere-defined operator A:*

(i) *There exists an operator L, with $\|L\|\,\|L^{-1}\| \leqq \varkappa$, such that $\|LAL^{-1}\| \leqq 1$;*

(ii) *For every rational f such that $|z| \leqq 1$ implies $|f(z)| \leqq 1$, we have $\|f(A)\| \leqq \varkappa$;*

(iii) *For every α with $|\alpha|<1$, $\|\mu_\alpha(A)\| \leqq \varkappa$;*

(iv) $s(A) \subseteqq L_\varkappa$.

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Leftrightarrow$(iv).

Let us assume (i), writing $A = L^{-1}CL$, $\|C\| \leqq 1$, $\|L\|\,\|L^{-1}\| \leqq \varkappa$. Then for every rational f without poles in the closed unit disk D, and in particular if $f(D) \subseteqq D$, we have $f(A) = L^{-1}f(C)L$ an everywhere-defined operator. If C is a contraction and $f(D) \subseteqq D$, von Neumann's theorem says $\|f(C)\| \leqq 1$; then it is obvious that $\|f(A)\| \leqq \varkappa$. In particular, f could be any of the μ_α. It has been proved that (i)$\Rightarrow$(ii)$\Rightarrow$(iii); note that the direct proof of (i)$\Rightarrow$(iii) is elementary. It remains to prove that (iii)$\Leftrightarrow$(iv).

If $s(A) \subseteqq L_\varkappa$ then every $(\zeta, h) \in s(A)$ satisfies $h \leqq \dfrac{-1+\varkappa^2}{1+\varkappa^2}$, that is, $\|A\| \leqq \varkappa$.

Now the μ_α, together with the multiplications by scalars of modulus 1 (transformations which do not affect norms of operators), generate under composition the group of all Möbius transformations of the unit disk onto itself, as is well known [11, V § 3]. Consider the transformations $\varrho(\mu)$, in the notation of § 5, as μ ranges over this group. They form a group of transformations of B onto itself, under which the equator is taken onto itself. If μ is multiplication by a scalar then it is obvious that $\varrho(\mu)L_\varkappa = L_\varkappa$, with the axis $\{(0, h)\}$ as fixpoints. We will see that $\varrho(\mu_\alpha)L_\varkappa = L_\varkappa$ and that this set of transformations acts transitively on the upper boundary

$$h = \frac{-1+\varkappa^2}{1+\varkappa^2}(1-|\zeta|^2)^{\frac{1}{2}}, \quad |\zeta| < 1. \tag{14. 1}$$

By Thm. 5. 1, $s(\mu(A)) = \varrho(\mu)s(A)$. It follows easily that $s(A) \subseteqq L_\varkappa$ if and only if, for all α with $|\alpha|<1$, $\|\mu_\alpha(A)\| \leqq \varkappa$. The only thing still required to establish the theorem is to verify the assertion about the action of $\varrho(\mu_\alpha)$ on the set (14. 1).

Refer again to § 5 for definition and properties of the δ-co-ordinates for points of B. It is easy to verify that $(\delta_1, \delta_2, \delta_3, \delta_4)$ are δ-co-ordinates of a point on the upper boundary (14. 1) if and only if they satisfy

$$\delta_1-\delta_4=\frac{-1+\varkappa^2}{1+\varkappa^2}\left((\delta_1+\delta_4)^2-4\delta_2\delta_3\right)^{\frac{1}{2}} \tag{14. 2}$$

in addition to the assumptions always made that $\delta_1=\overline{\delta_1}$, $\delta_4=\overline{\delta_4}$, $\delta_2=\overline{\delta_3}$, $\delta_2\delta_3\leqq\leqq\delta_1\delta_4$. Also from § 5, $\varrho(\mu_\alpha)$ transforms the δ-co-ordinates by the matrix

$$\begin{pmatrix} 1 & -\bar{\alpha} & -\alpha & |\alpha|^2 \\ -\alpha & 1 & \alpha^2 & -\alpha \\ -\bar{\alpha} & \bar{\alpha}^2 & 1 & -\bar{\alpha} \\ |\alpha|^2 & -\bar{\alpha} & -\alpha & 1 \end{pmatrix}. \tag{14. 3}$$

I leave to the reader the routine verification that (14. 2) is invariant under the change of co-ordinates (14. 3). This makes it clear that $\varrho(\mu_\alpha)L_\varkappa=L_\varkappa$. It is also easy to see from (14. 3) how to choose that α such that $\varrho(\mu_\alpha)$ will take a given (ζ, h) on the upper boundary to the top point $\left(0, \dfrac{-1+\varkappa^2}{1+\varkappa^2}\right)$. One gets the equation $\alpha^{-1}\zeta+\alpha\bar{\zeta}=2$, and for $|\zeta|<1$ this does have a root with $|\alpha|<1$. This establishes transitivity, and completes the proof of the theorem.

Note that for $\varkappa=1$, (i) says that A itself is a contraction, so (i)$\Leftrightarrow$(iv) for this case. The equivalence fails for higher $\varkappa$. Indeed, consider the Foguel—Halmos example [7] of a power-bounded operator. As A. L. Shields pointed out to me, it is easy to see that it satisfies (iii) with $\varkappa=6$, and on the other hand it is known that it does not satisfy (ii) for any $\varkappa$. Also, as was pointed out to me by Sz.-Nagy and Foiaş, the Cayley transform of the example of Markus [10, § 4] is relevant here. It satisfies (iii) but does not satisfy (i) for any $\varkappa$; it is not clear whether it can be adjusted so as to satisfy (ii).

Nevertheless, $L_\varkappa$ cannot be replaced by any proper subset in the statement of Thm. 14. 1. For consider again the operator $A=\begin{pmatrix}0 & 1/\varkappa \\ \varkappa & 0\end{pmatrix}$. It does not belong to any of the classes C_ϱ, because its shell is an ellipsoid which is tangent to the unit sphere S at equatorial points, and hence is not a subset of any K_ϱ or E_ϱ (see § 13). A does obviously satisfy (i), with the same value of $\varkappa$; and the same applies to $e^{i\delta}A$ for any real δ. But $\bigcup_\delta s(e^{i\delta}A)$ gives all of $L_\varkappa$ above the equatorial plane.

To Thm. 14. 1, which concerns the union of shells of operators similar to contractions, we may contrast this theorem about the intersections of the shells of operators similar to a fixed operator:

Theorem 14. 2. *Assume* $\dim\mathfrak{H}\geqq 3$. *Then* $\operatorname{conv}(\tau(\sigma_\pi(A)))=\bigcap\overline{s(T^{-1}AT)}$, *the intersection being over all invertible* T.

Proof. Since $\sigma_\pi(T^{-1}AT)=\sigma_\pi(A)$, Thm. 2. 3 tells us that $\tau(\sigma_\pi(A))\subseteq\overline{s(T^{-1}AT)}$; then Thm. 10. 2 tells us that the same holds for the convex hull. This yields inclusion in one direction.

For the converse, let p be any point of B not in $\operatorname{conv}(\tau(\sigma_\pi(A)))$; and let π be any plane strictly separating p from $\tau(\sigma_\pi(A))$. We will be done if we prove that, for some T, $s(T^{-1}AT)$ lies entirely in one of the closed half-spaces determined by π (necessarily, that closed half-space containing $\tau(\sigma_\pi(A))$).

Let q be any point of S which lies neither in $\tau(\sigma(A))$ nor in π. Applying a Möbius transformation, we can bring π to the equatorial plane, and in such a way that q goes to the north pole; see § 5. Since we chose q not to be in $\tau(\sigma(A))$, the Möbius transform of A is still an operator and not just a relation, and the same applies to each $T^{-1}AT$. Thus there is no loss of generality in assuming that π was the equatorial plane to begin with. Now there are two cases, depending on whether p and q are on the same side of π.

Case I. Under the hypothesis that $\sigma_\pi(A)$ lies entirely in the open unit disc, we are to prove that, for some T, $\|T^{-1}AT\|\leqq 1$. The following key argument is taken from S. HILDEBRANDT's proof of the corresponding theorem for numerical ranges [9, Lemma 1 and Satz 4]. Because the spectral radius of A is <1, we know $\sum_0^\infty A^{*n}A^n$ converges in norm. Its limit is a self-adjoint operator H, which, because it is $\geqq 1$, must be invertible. Clearly $A^*HA = H-1\leqq H$. Setting $T=H^{-\frac{1}{2}}$, we verify that $\|T^{-1}AT\|\leqq 1$ as follows: for any $x\in\mathfrak{H}$, write $x=T^{-1}y$, then

$$\|T^{-1}ATx\|^2 = \|T^{-1}Ay\|^2 = y^*A^*HAy \leqq y^*Hy = \|T^{-1}y\|^2 = \|x\|^2.$$

Case II. Under the hypothesis that $\sigma_\pi(A)$ lies entirely outside the closed unit disc, we are to prove that, for some T, $\|T^{-1}ATx\|\geqq\|x\|$ for all x. The idea of Hildebrandt can be modified as follows. Let B be the pseudo-inverse of A; that is, B restricted to $\mathfrak{R}(A)$ is A^{-1}, while B restricted to $\mathfrak{R}(A)^\perp$ is 0. Then B is an everywhere-defined bounded operator, and $BA=1$ because $\mathfrak{N}(A)=\{0\}$. Here $\sigma_\pi(B)\subseteq\subseteq(\sigma_\pi(A))^{-1}\cup\{0\}$; one way to see this is from Prop. 6. 4 and Prop. 3. 1. Thus the spectral radius of B is <1, and this time we set $H=\sum_0^\infty B^{*n}B^n\geqq 1$. Clearly $A^*HA= = A^*A+H\geqq H$. As before, set $T=H^{-\frac{1}{2}}$, and compute for any $x=T^{-1}y$,

$$\|T^{-1}ATx\|^2 = \|T^{-1}Ay\|^2 = y^*A^*HAy \geqq y^*Hy = \|T^{-1}y\|^2 = \|x\|^2.$$

This completes the proof.

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