Unitary dilations and coisometric extensions

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Let $\mathfrak{H}$ be a complex Hilbert space, and let $\mathfrak{B}(\mathfrak{H})$ denote the algebra of bounded linear operators on $\mathfrak{H}$. For a subspace $\mathfrak{M}\subset\mathfrak{H}$, let $P_{\mathfrak{M}}$ denote the projection of $\mathfrak{H}$ onto $\mathfrak{M}$. If S is a commutative semigroup with identity 0, then we say that $\mathcal{T}=\{T(g): g\in S\}$ is a *semigroup of operators* on $\mathfrak{H}$ if $T(g)\in\mathfrak{B}(\mathfrak{H})$, $T(0)=I$, and $T(g_1+g_2) = T(g_1)T(g_2)$ for all $g_1, g_2\in S$. We write $\mathcal{T}^*=\{T(g)^*: g\in S\}$. A semigroup $\mathcal{D}$ of operators on $\mathfrak{K}\supset\mathfrak{H}$ is called a *dilation* of $\mathcal{T}$ if $T(g)=P_{\mathfrak{H}}D(g)|\mathfrak{H}$ for all $g\in S$, while $\mathcal{D}$ is called an *extension* of $\mathcal{T}$ if $T(g)=D(g)|\mathfrak{H}$ for all $g\in S$; here $\mathfrak{H}$ must be invariant for $D(g)$.

We first prove a theorem relating unitary dilations and coisometric extensions. The proof is an extension of a proof by SZ.-NAGY—FOIAŞ [3, p. 12] for the semigroup $S=Z^+$ (the additive semigroup of non-negative integers), and uses the following theorem of ITO [2]: Every isometric semigroup has a unitary extension.

Theorem 1. *A semigroup $\mathcal{T}=\{T(g): g\in S\}$ of operators on $\mathfrak{H}$ has a unitary dilation if and only if it has a coisometric extension.*

Proof. Let $\mathcal{V}^*$ on $\mathfrak{K}_1\supset\mathfrak{H}$ be a coisometric extension of $\mathcal{T}$. By ITO's theorem $\mathcal{V}$ has a unitary extension $\mathcal{U}^*$ on $\mathfrak{K}_2\supset\mathfrak{K}_1$. But $V(g)=U(g)^*|\mathfrak{K}_1$ implies $V(g)^*= =P_{\mathfrak{K}_1}U(g)^*|\mathfrak{K}_1$. Hence

$$T(g) = V(g)^*|\mathfrak{H} = P_{\mathfrak{H}}(P_{\mathfrak{K}_1}U(g)|\mathfrak{K}_1)|\mathfrak{H} = P_{\mathfrak{H}}U(g)|\mathfrak{H}.$$

So $\mathcal{U}$ is a unitary dilation of $\mathcal{T}$.

Conversely, let $\mathcal{U}$ on $\mathfrak{K}_2\supset\mathfrak{H}$ be a unitary dilation of $\mathcal{T}$. Define $\mathcal{V}$ on

$$\mathfrak{K}_1 = \bigvee_{g\in S} U(g)^*\mathfrak{H}$$

by

$$V(g) = U(g)^*|\mathfrak{K}_1;$$

$\mathfrak{K}_1$ is invariant for $\mathcal{U}^*$ and includes $\mathfrak{H}$. Hence

$$T(g) = P_{\mathfrak{H}}U(g)|_{\mathfrak{H}} = P_{\mathfrak{H}}(P_{\mathfrak{K}_1}U(g)|\mathfrak{K}_1)|\mathfrak{H} = P_{\mathfrak{H}}V(g)^*|\mathfrak{H}.$$

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In order to prove that $\mathscr{V}^*$ is an extension of $\mathscr{T}$ we need only show that $\mathfrak{H}$ is invariant for $\mathscr{V}^*$. Let Q be the projection onto $\mathfrak{K}_1 \ominus \mathfrak{H}$. If $x \in \mathfrak{H}$, then

$$V(g)x = T(g)^*x + \big(V(g) - T(g)^*\big)x.$$

Hence

$$QV(g)x = \big(V(g) - T(g)^*\big)x.$$

Since

$$V(g_1)(V(g) - T(g)^*) = (V(g_1+g) - T(g_1+g)^*) - (V(g_1) - T(g_1)^*)T(g)^*$$

and because

$$\mathfrak{K}_1 \ominus \mathfrak{H} = Q\mathfrak{K}_1 = \bigvee_{g\in S} QV(g)\mathfrak{H} = \bigvee_{g\in S} \big(V(g) - T(g)^*\big)\mathfrak{H},$$

$\mathfrak{K}_1 \ominus \mathfrak{H}$ is invariant for $\mathscr{V}$, or equivalently, $\mathfrak{H}$ is invariant for $\mathscr{V}^*$.

For a set A, we now consider the special semigroup

$$S = Z^{+(A)} \equiv \{\text{finitely non-zero functions from } A \text{ to } Z^+\}.$$

For $g \in S$, $\operatorname{supp}(g) \equiv \{\omega \in A : g(\omega) \neq 0\}$ is a finite set. For v a finite subset of A, let $\chi_v \in S$ be defined by:

$$\chi_v(\omega) = 1 \quad \text{if} \quad \omega \in v, \quad \text{and} = 0 \quad \text{otherwise},$$

and let $|v| =$ the number of elements of v. If $\mathscr{T} = \{T(g): g \in S\}$ is a semigroup of operators, we write $T_\omega = T(\chi_{\{\omega\}})$ for $\omega \in A$. We say that $\mathscr{T}$ is a **-commuting* semigroup of operators, if $T_\omega T_\lambda^* = T_\lambda^* T_\omega$ for all $\omega, \lambda \in A$, $\omega \neq \lambda$.

Our next theorem relates a particular kind of unitary dilation to an extension to a particular kind of coisometric semigroup. The former is called a Sz.-Nagy—Brehmer dilation [1] or a regular dilation [3] in the literature. It is natural to call the latter a *-commuting coisometric extension.

Theorem 2. *If $\mathscr{T}$ is a semigroup of operators on H with $S = Z^{+(A)}$, then the following are equivalent:*

(i) *there exists a *-commuting coisometric extension of $\mathscr{T}$;*

(ii) *there exists a unitary dilation $\mathscr{U}$ of $\mathscr{T}$ satisfying*

$$T(g_1)^*\, T(g_2) = P_H U(g_2) U(g_1)^* | H$$

for $g_1, g_2 \in S$ with disjoint supports;

(iii) *for all finite subsets F of A,* $\sum_{v \subset F} (-1)^{|v|} T(\chi_v)^* T(\chi_v) \geqq 0.$

Proof. That (ii) and (iii) are equivalent is Theorem 9. 1 in Sz.-Nagy and Foiaş [3]. We will prove that (i) and (ii) are equivalent.

Let $\mathscr{V}^*$ be a *-commuting coisometric extension on $\mathfrak{K}_1 \supset \mathfrak{H}$ of $\mathscr{T}$. Then by Ito's theorem there exists a unitary extension $\mathscr{U}^*$ on $\mathfrak{K}_2 \supset \mathfrak{K}_1$ of $\mathscr{V}$. If $g_1, g_2 \in S$ have disjoint supports, then

$$\begin{aligned} T(g_1)^* T(g_2) &= P_{\mathfrak{H}} V(g_1)|\mathfrak{H} \cdot V(g_2)^*|\mathfrak{H} = P_{\mathfrak{H}} V(g_1) V(g_2)^*|\mathfrak{H} \\ &= P_{\mathfrak{H}} V(g_2)^* V(g_1)|\mathfrak{H} \quad \text{by *-commutativity} \\ &= P_{\mathfrak{H}} [P_{\mathfrak{K}_1} U(g_2)|\mathfrak{K}_1 \cdot U(g_1)^*|\mathfrak{K}_1]|\mathfrak{H} = P_{\mathfrak{H}} U(g_2) U(g_1)^*|\mathfrak{H}, \end{aligned}$$

so that $\mathscr{U}$ is a unitary dilation of $\mathscr{T}$ satisfying (ii).

On the other hand, let $\mathscr{U}$ on $\mathfrak{K}_2 \supset \mathfrak{H}$ be a unitary dilation of $\mathscr{T}$, which satisfies (ii). As in the proof of Theorem 1, if

(1) $$\mathfrak{K}_1 = \bigvee_{g \in S} U(g)^* \mathfrak{H} \quad \text{and} \quad V(g) = U(g)^*|\mathfrak{K}_1$$

then

(2) $$T(g) = V(g)^*|\mathfrak{H}.$$

Thus $\mathscr{V}^*$ on $\mathfrak{K}_1 \supset \mathfrak{H}$ is a coisometric extension of $\mathscr{T}$. We need only show that $\mathscr{V}^*$ is a *-commuting semigroup.

If $g_1, g_2 \in S$ have disjoint supports, then, by (ii) and (1),

(3) $$T(g_1)^* T(g_2) = P_{\mathfrak{H}} U(g_2) U(g_1)^*|\mathfrak{H} = P_{\mathfrak{H}} V(g_2)^* V(g_1)|\mathfrak{H}.$$

Also, by (2)

(4) $$T(g_1)^* T(g_2) = P_{\mathfrak{H}} V(g_1)|\mathfrak{H} \cdot V(g_2)^*|\mathfrak{H} = P_{\mathfrak{H}} V(g_1) V(g_2)^*|\mathfrak{H}.$$

Subtracting (3) from (4) we obtain

(5) $$P_{\mathfrak{H}}\big(V(g_1)V(g_2)^* - V(g_2)^* V(g_1)\big)|\mathfrak{H} = 0,$$

for $g_1, g_2 \in S$ with disjoint supports.

We claim that for all $g \in S$

(6) $$P_{\mathfrak{H}} V(g)^*\big(V(g_1)V(g_2)^* - V(g_2)^* V(g_1)\big)|\mathfrak{H} = 0.$$

We first remark that it is sufficient to prove (6) for g such that g and g_1 have disjoint supports. (Note that $V(g)^* V(g_1) = V(g')^* V(g_1')$ where g' and g_1' have disjoint supports. In fact take $g' = g - \min(g, g_1)$ and $g_1' = g_1 - \min(g, g_1)$.) Let $g \in S$ be such that $\operatorname{supp}(g)$ is disjoint from $\operatorname{supp}(g_1)$. Then $\operatorname{supp}(g+g_2)$ is disjoint from $\operatorname{supp}(g_1)$, so that (5) implies

(7) $$P_{\mathfrak{H}}\big(V(g_1)V(g+g_2)^* - V(g+g_2)^* V(g_1)\big)|\mathfrak{H} = 0,$$

and

(8) $$P_{\mathfrak{H}}\big(V(g_1)V(g)^* - V(g)^* V(g_1)\big)|\mathfrak{H} = 0.$$

Multiplying in (8) by $V(g_2)^*|\mathfrak{H}$ from the right, and subtracting from (7), we obtain (6).

Since $\mathscr{V}^*$ acts on

$$\mathfrak{K}_1 = \bigvee_{g \in S} V(g)\mathfrak{H}$$

and since $P_{V(g)\mathfrak{H}} = V(g)P_{\mathfrak{H}}V(g)^*$, (6) implies

(9) $$(V(g_1)V(g_2)^* - V(g_2)^*V(g_1))|\mathfrak{H} = 0,$$

for $g_1, g_2 \in S$ with disjoint supports.

We now claim that for all $g \in S$

(10) $$(V(g_1)V(g_2)^* - V(g_2)^*V(g_1))V(g)|\mathfrak{H} = 0.$$

As above we need only prove (10) for $g \in S$ such that g and g_2 have disjoint supports. We use (9) to obtain

(11) $$(V(g_1+g)V(g_2)^* - V(g_2)^*V(g_1+g))|\mathfrak{H} = 0,$$

and

(12) $$(V(g)V(g_2)^* - V(g_2)^*V(g))|\mathfrak{H} = 0.$$

Multiplying in (12) by $V(g_1)$ from the left and substracting from (11) we obtain (10).

But (10) implies

$$V(g_1)V(g_2)^* - V(g_2)^*V(g_1) = 0,$$

for $g_1, g_2 \in S$ with disjoint supports. In particular for $g_1 = \chi_{\{\omega\}}$, $g_2 = \chi_{\{\lambda\}}$ with $\omega, \lambda \in A$, $\omega \neq \lambda$. Thus $\mathscr{V}^*$ is a *-commuting coisometric semigroup.

Bibliography

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