On operator representations of function algebras

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In this note we are going to study contractive representations of function algebras. We shall determine the structure of the representations of a special class of function algebras. The final result is a generalization of those given by D. SARASON in [8].

§ 1. Introduction

1. Let X be a compact Hausdorff space, $C(X)$ the algebra of all continuous complex functions on X, and A a function algebra on X (i.e. a closed subalgebra of $C(X)$ which contains the constants and separates the points of X). Denote by $M_A = M$ the maximal ideal space, by $\hat{A}$ the image of the Gelfand representation of A. Suppose that each complex homomorphism of A has a unique representing measure. Fix θ in the Gleason part $\mathscr{P}$ of M, and denote by m its representing measure.

We write A_0 for the kernel of θ. $H^\infty(m)$, $H_0^\infty(m)$ are defined (as usual) as the w^*-closure of A, A_0, respectively, in $L^\infty(m)$. For $1 \leqq p < \infty$, $H^p(m)$, $H_0^p(m)$ are also defined as the closures of A, A_0 in $L^p(m)$, respectively.

Let Y be the space of the maximal ideals of $L^\infty(m)$, and identify $L^\infty(m)$ with $C(Y)$ via the Gelfand representation. Under this representation, $H^\infty(m)$ is mapped onto a subalgebra $\hat{H}^\infty(m)$ of $C(Y)$. $\hat{H}^\infty(m)$ is logmodular on Y, and consequentely its Shilov boundary is exactly Y. It is known*) that each $\chi \in \mathscr{P}$ can be uniquely extended to a continuous linear functional $\hat{\chi}$ on $H^2(m)$, multiplicative on $H^\infty(m)$. Also $\hat{\mathscr{P}} = \{\hat{\chi}, \chi \in \mathscr{P}\}$ is the Gleason part of $H^\infty(m)$ containing $\hat{\theta}$. Denote by $\hat{m}$ the representing measure (which is supported on Y) for $\hat{\theta}$.

The following result will play an important role in the sequel:

If $\mathscr{P}$ is a non-trivial Gleason part of M, i.e. it consists of more than one point, then there exists an inner function $Z \in H^\infty(m)$ such that $H_0^\infty(m) = ZH^\infty(m)$. (See [2], chap. VI th. 7. 1. and 7. 2.)

We shall need the following:

*) For references of this section see for instance [2].

Proposition 1. *The following are equivalent for a non-trivial Gleason part $\mathcal{P}$:*

(a) *The map* $\tau\colon g \to \sum_{0}^{\infty} a_n \lambda^n$, *where* $a_n = \int g\bar{Z}^n dm$, *is an isometric isomorphism of* $H^\infty(m)$ *onto the Hardy algebra* H^∞.

(b) *If J is the ideal of all* $g \in H^\infty(m)$ *with* $\hat{g}(\chi) = 0$ *for* $\chi \in \mathcal{P}$, *then* $J = 0$.

(c) *The linear span of* $\{Z^n\}_0^\infty$ *is* w^**-dense in* $H^\infty(m)$.

Proof. This follows easily from [2], exc. 6. p. 141 and from discussion on p. 163 of MERRILL's paper [4].

We note that there exist algebras for which the conditions of proposition 1 hold true. (See [4], §§ 2, 3.)

Finally we note that for the function Z the following theorem holds:

If there is a uniformly closed algebra B in $L^\infty(m)$ strictly containing $H^\infty(m)$, then $Z^{-1} \in B$.

The proof is similar to that of the theorem of section "Maximality" in [3], chap. 10, and makes use of the relation:

$$L^2(m) = H^2(m) \oplus \overline{H_0^2(m)}.$$

2. We start this section with giving definitions concerning representations of function algebras. (See [1], [5].)

Let A be a function algebra on X, and H a complex Hilbert space. By definition a *contractive representation* of A on H is an algebra homomorphism $f \to T_f$ of A into the algebra $\mathcal{B}(H)$ of all bounded linear operators on H such that T_1 is the identity operator and

$$(*) \qquad \|T_f\| \leqq \|f\| \qquad (f \in A).$$

The representation $f \to T_f$ of A on H is called *X-reducing* if it is the restriction to A of a representation of $C(X)$ on H. A subspace of H is called *X-reducing* if it reduces the representation $f \to T_f$ to an X-reducing one. The representation $f \to T_f$ of A on H is called *X-pure* if $\{0\}$ is the only X-reducing subspace for it. The representation $f \to T_f$ is called *X-dilatable* if there exists a representation $\varphi \to U_\varphi$ of $C(X)$ on K such that K contains H as a subspace, and

$$T_f x = P U_f x \qquad (f \in A;\ x \in H),$$

where P is the orthogonal projection of K onto H; the representation $\varphi \to U_\varphi$ is then called the *X-dilation* of the representation $f \to T_f$. A contractive representation of A on H is *X-maximal*, if there is no algebra B distinct from A and $C(X)$, satisfying $A \subset B \subset C(X)$ and having a representation on H which coincides on A with the initial one.

Now by the theorems of Hahn—Banach and Riesz—Kakutani, we observe that there exists a family $\{p_{x,y}\}$ of measures on X such that

$$(T_f x, y) = \int f \, dp_{x,y} \qquad (f \in A;\ x, y \in H).$$

Such a family $\{p_{x,y}\}$ is called *a family of elementary measures*. By definition the representation $f \to T_f$ is called *$\mathscr{P}$-continuous* if there exists a family $\{p_{x,y}\}$ of elementary measures such that $p_{x,y}$ is, for all $x, y \in H$, absolutely continuous with respect to m. If there exists a family $\{p_{x,y}\}$ of elementary measures such that $p_{x,y}$ is, for all $x, y \in H$, singular with respect to all representing measures for A, then the representation is called *completely singular*.

A $\mathscr{P}$-continuous representation $f \to T_f$ of A on H has the following properties (see [5]):

(i) *$f \to T_f$ has a unique extension to a representation $g \to \hat{T}_g$ of $H^\infty(m)$ on H; here $(*)$ is to be replaced by*

$$(*') \qquad \|\hat{T}_g\| \leqq \|g\|_\infty = \|\hat{g}\| \qquad (g \in H^\infty(m)).$$

(ii) *There is a unique semi-spectral measure F on X, absolutely continuous with respect to m, such that*

$$\hat{T}_g = \int g \, dF \qquad (g \in H^\infty(m)).$$

Therefore by Neumark's theorem $f \to T_f$ is an X-dilatable representation.

(iii) *The extension $g \to \hat{T}_g$ is $\hat{\mathscr{P}}$-continuous, hence there exists a semi-spectral measure $\hat{F}$ on Y, absolutely continuous with respect to $\hat{m}$, such that*

$$\hat{T}_g = \int \hat{g} \, d\hat{F} \qquad (g \in H^\infty(m)).$$

We also mention the following supplementary properties:

(iv) *If the representation $f \to T_f$ is X-pure, then the extended representation $g \to \hat{T}_g$ is Y-pure.*

(v) *This extension is Y-maximal if and only if $\hat{T}_Z$ is a non-unitary contraction.*

Proof. Ad (iv). We suppose that H^u is a Y-reducing subspace for $g \to \hat{T}_g$. Then $g \to \hat{T}^u_g = \hat{T}_g|_{H^u}$ is an Y-reducing representation, and for the continuous functions on X we have:

$$\|\hat{T}^u_\varphi\| \leqq \|\varphi\|_\infty \leqq \|\varphi\|;$$

so H^u is an X-reducing subspace for the initial representation as well, and this is a contradiction.

Ad (v). If the representation $g \to \hat{T}_g$ of $H^\infty(m)$ is not Y-maximal, then there is an extension of the initial representation to a subalgebra of $L^\infty(m)$ containing $H^\infty(m)$. Then by the final remark of section 1, $\hat{T}_{Z^{-1}}$ makes sense and the relation

$\hat{T}_Z\hat{T}_{Z^{-1}}=\hat{T}_{Z^{-1}}\hat{T}_Z=I$ contradicts the non-unitarity of $\hat{T}_Z$. Conversely, if the representation $g\to\hat{T}_g$ is Y-maximal, then $\hat{T}_Z$ is non-unitary, since otherwise there would be an obvious extension of the representation $g\to\hat{T}_g$ $(g\in H^\infty(m))$ to the algebra generated by $H^\infty(m)$ and Z^{-1}, implying $\hat{T}_{Z^{-1}}=\hat{T}_Z^*$.

§ 2. The structure of the representations for some function algebras

1. Let D be the open unit disc, and $\mathscr{A}$ the algebra consisting of those functions continuous in the closure $\bar{D}$ of D which are analytic on D. It is known that D is the Gleason part containing the 0-homomorphism, and the (unique) representing measure for 0 is the normalized Lebesgue measure μ on the unit circle ∂D (which is the Shilov boundary of $\mathscr{A}$). The algebra $H^\infty(\mu)$ coincides with the Hardy algebra H^∞ if we identify each function of H^∞ with its boundary values. (See [2], chap. II, sec. 4). Denote by Γ the maximal ideal space of $L^\infty(\mu)$.

In this section we prove:

Theorem 1. *Let $g\to T_g$ be a contractive $\hat{D}$-continuous representation of H^∞ on a Hilbert space H. Then:*

(a) *If $T=T_{e^{it}}$, then $H_T^\infty=H^\infty$ and*

$$T_g=g(T)\qquad(g\in H^\infty),$$

where the right-hand term is defined by the functional calculus of [7] *chap. III.*

(b) *If the representation $g\to T_g$ is Γ-pure, then T is a completely non-unitary contraction on H.*

Proof. (a) It is easy to prove that the restriction of the initial representation to $\mathscr{A}$ is a contractive representation of $\mathscr{A}$ which is D-continuous. Then, by (ii), there exists a semi-spectral measure E on ∂D, absolutely continuous with respect to μ, such that

$$(T_gx,y)=\int g\,d(Ex,y)\quad(g\in H^\infty;\ x,y\in H).$$

Now by an easy computation we deduce:

$$T_{g_r}\to T_g\quad\text{strongly as}\quad r\to 1,\tag{1}$$

where $g_r(e^{it})=g(re^{it})$, $g\in H^\infty$. Since the Fourier series $\sum_0^\infty a_ne^{int}$ of $g_r(e^{it})$ is absolutely and uniformly convergent, it is immediate that

$$T_{g_r}=\sum_0^\infty a_nT^n=g_r(T).\tag{2}$$

Then from the definition of H_T^∞ *) it follows that $g \in H_T^\infty$, and by (1) and (2) we obtain

$$T_g = g(T) \qquad (g \in H^\infty).$$

Point (b) results easily from (a) and by the remark that the Γ-purity of the representation $g \to T_g$ implies the ∂D-purity of its restriction to $\mathscr{A}$. Now, the ∂D-purity of the representation $f \to f(T)$ ($f \in \mathscr{A}$) is equivalent to the fact that T is a completely non-unitary contraction (See [8]).

2. Now we return to an arbitrary function algebra A. The notations and the hypothesis given in sec. 1, § 1 are the same. In the following we assume that the conditions of proposition 1 hold true for the non-trivial Gleason part $\mathscr{P}$. If τ^* is the adjoint of the isometry τ of proposition 1, then it is immediate that $\tau^*(M_{H^\infty}) = M_{H^\infty(m)}$, and $\tau^*(\Gamma) = Y$. Consequently τ can be extended to an isometric isomorphism of $L^\infty(m)$ onto $L^\infty(\mu)$. It is also clear that $\tau^*(\hat{D}) = \hat{\mathscr{P}}$.

Now let $f \to T_f$ be a (contractive) representation of $H^\infty(m)$ on H. Then it is obvious that the map: $g \to S_g = T_f$, $\tau f = g$ is a representation of H^∞ on H. The following lemma is almost immediate.

Lemma 1. (a) *If the representation $f \to T_f$ is Y-pure, then the representation $g \to S_g$ is Γ-pure.*

(b) *If the representation $f \to T_f$ is $\hat{\mathscr{P}}$-continuous, then $g \to S_g$ is $\hat{D}$-continuous. Moreover if $\hat{F}$ is the semi-spectral measure of the representation $f \to T_f$, then $\tau^* \hat{F}$ is the semi-spectral measure for the representation $g \to S_g$.*

We are now able to prove our main result.

Theorem 2. *Let $f \to T_f$ be a $\mathscr{P}$-continuous representation of A on the Hilbert space H, and $g \to \hat{T}_g$ the extended representation given in* (i). *If $T = \hat{T}_Z$, then for each $g \in H^\infty(m)$ we have $\tau g \in H_T^\infty$ and*

$$T_g = (\tau g)(T) \qquad (g \in H^\infty(m)).$$

If the representation is X-pure, then:

(a) *T is a completely non-unitary contraction on H;*

(b) *the extended representation $g \to \hat{T}_g$ is Y-maximal;*

(c) *the semi-spectral measure F, defined by* (ii), *is equivalent to m;*

(d) *for each $x \in H$, $x \neq 0$, the logarithm of the Radon-Nikodym derivative of $(E(\cdot)x, x)$ with respect to m is in $L^1(m)$.*

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*) For the definition of H_T^∞ see [7] chap. III.

Proof. The first statement follows by combining (iii) and Lemma 1 with point (a) of Theorem 1. Since $\tau Z=e^{it}$, point (a) follows by using Lemma 1 and point (b) of Theorem 1. By (v), (a) implies (b). For (c), it remains only to prove that m is absolutely continuous with respect to F. If E is the semi-spectral measure (on ∂D) of T, one can prove easily (by using the uniqueness of elementary measures $\{(Fx, y)\}$ and point (b) of Lemma 1) that

$$\int \psi\, d(Fx, y) = \int \tau\psi\, d(Ex, y) \qquad (\psi \in L^\infty(m);\ x, y \in H). \tag{3}$$

Now by (3) and by Szegő's theorem we deduce

$$\exp \int \log \frac{d(Fx, x)}{dm}\, dm = \exp \int \log \frac{d(Ex, x)}{d\mu}\, d\mu.$$

Therefore by Proposition 6. 5, chap. II of [7] concerning μ-summability of $\log \frac{d(Fx,x)}{d\mu}$, (d) follows. Now (d) implies in particular that m is absolutely continuous with respect to F. This completes the proof.

3. Point (d) in the preceding theorem is due to W. MLAK.*) Moreover W. MLAK indicated us the following direct proof of (a), (c), (d).

Let $f \to T_f$ be a contractive $\mathscr{P}$-continuous representation of A on H. According to Theorem 1 of [1], the representation $f \to T_f$ is a uniquely determined orthogonal sum of an X-reducing representation $f \to T_f^u$ and an X-pure representation $f \to T_f^0$, corresponding to the decomposition $H^u \oplus H^0$ of H. On the other hand, by (ii), the representation $f \to T_f$ is X-dilatable. Denote by $\varphi \to U_\varphi$ this dilation, and by K the dilation space. We prove the following lemma.

Lemma 2. *If for some $x \in H$:*

$$\inf_{A_0} \int |1-f|^2\, d(Fx, x) = 0, \tag{γ}$$

then $W = \bigvee_{-\infty}^{\infty} U^n x \subset H^u$. (Here U means $\hat{U}_Z$).

Proof. First we prove that (γ) implies

$$\|T^n x\| = \|x\| = \|T^{*n} x\| \qquad (n = 0, 1, 2, \ldots). \tag{δ}$$

If $f \in A_0 \subset H_0^\infty(m)$, then $f = Zg$, where $g \in H^\infty(m)$. Let $R = \bigvee_0^\infty U^n x$, and let Q be the orthogonal projection of K on R. Then

$$((I-Q)U_g x, y) = (U_g x - s, (I-Q)y) \qquad (s \in R, y \in K).$$

*) Private communication.

Let $\varepsilon>0$. By (c) of Proposition 1 there is an $s_\varepsilon=\sum_{k\geqq 0} a_k U^k x$ (finite sum) such that:

$$|(U_g x-s_\varepsilon, (I-Q)y)| = \left|\int \left(g-\sum_{k\geqq 0} a_k Z^k\right) h\, dm\right| < \varepsilon \quad \left(h=\frac{d(Fx, (I-Q)y)}{dm}\right).$$

It follows that $U_g x\in R$. Consequentely $x\in \bigvee_1^\infty U^n x$, by ($\gamma$). Now by Lemma 6. 4, p. 44 of [6] and by obvious symmetry (δ) holds true. To obtain $W\subset H^u$, it is sufficient to prove (by (δ)) that for $z=U_f x$ ($f\in A$) we have $(I-P)z=0$. (Here P is the orthogonal projection of K onto H.) For $s\in H$ we have, for any $y\in K$, $((I-P)z, y) = ((z-s), (I-P)y)$. Let $\varepsilon>0$. By (c) of Proposition 1, there is an $s_\varepsilon=\sum_{k\geqq 0} a_k U^k x$ (finite sum), which by (δ) is contained in H, such that

$$|(z-s_\varepsilon, (I-P)y)| = \left|\int \left(f-\sum_{k\geqq 0} a_k Z^k\right) h\, dm\right| < \varepsilon \quad \left(h=\frac{d(Fx, (I-P)y)}{dm}\right).$$

It follows by the arbitrariness of y that $z=Pz$. By symmetry we have $U_f x\in H$, which completes the proof.

Now, if the representation $f\to T_f$ is X-pure, then Lemma 2 implies that $T=\hat{T}_Z$ is c.n.u. and using Szegő's theorem we get that $\log\frac{d(Fx,x)}{dm}\in L^1(m)$ for any $x\in H$, $x\neq 0$. This last property shows that m is absolutely continuous with respect to F.

4. Let $\{\mathscr{P}_\alpha\}_{\alpha\in I}$ be the set of all Gleason parts of the function algebra A, and $\{\mathscr{P}_\alpha\}_{\alpha\in I_0}$ the set of those non-trivial Gleason parts for which the conditions of Proposition 1 hold true. If $\alpha\in I_0$, then we denote by τ_α the isomorphism introduced in Proposition 1, corresponding to the Gleason part $\mathscr{P}_\alpha$.

Let $f\to T_f$ be a contractive representation of A on H, and

$$T_f = T_f^u \oplus T_f^0 \qquad (f\in A)$$

its decomposition given by Theorem 1 of [1]. To the X-pure representation $f\to T_f^0$ we apply the Mlak's decomposition (see [5]):

$$T_f^0 = \bigoplus_{\alpha\in I} T_f^\alpha \oplus T_f^c \qquad (f\in A),$$

where $f\to T_f^\alpha$ is the $\mathscr{P}_\alpha$-continuous part ($\alpha\in I$), and $f\to T_f^c$ is the comletely singular part of the representation $f\to T_f^0$. Now for the $\mathscr{P}_\alpha$-continuous parts we apply the Theorem 2 and finaly we obtain:

Theorem 3. *A contractive representation $f\to T_f$ of A on H is a uniquely determined orthogonal sum of the form:*

$$T_f = T_f^u \oplus \bigoplus_{\alpha\in I_0} (\tau_\alpha f)(T_\alpha) \oplus \bigoplus_{\alpha\notin I_0} T_f^\alpha \oplus T_f^c \qquad (f\in A),$$

where $f \to T_f^u$ *is an X-reducing representation,* T_α *are completely non-unitary contractions* $(\alpha \in I_0)$, $f \to T_f^\alpha$ $(\alpha \notin I_0)$ *are* $\mathscr{P}_\alpha$*-continuous and X-pure representations, and* $f \to T_f^c$ *is an X-pure and completely singular representation.*

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