On the stability of the zero solution of certain second order non-linear differential equations

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Introduction

In this paper we shall study the equation

$$\text{(E)} \qquad x'' + a(t)g(x, x')x' + b(t)f(x) = 0$$

under the following assumptions:

(A_1) $a(t) \in C[0, \infty)$, $a(t) \geqq 0$; $b(t) \in C'[0, \infty)$, $b(t) > 0$;

(A_2) $f(u) \in C(-\infty, \infty)$, $uf(u) > 0$ $(u \neq 0)$, and $\lim_{|u| \to \infty} F(u) = \infty$, where

$$F(u) = \int_0^u f(x)\, dx;$$

(A_3) $g(u, v)$ is continuous and non-negative on the (u, v) plane;

(A_4) for arbitrary $t_0 \geqq 0$, x_0, x_0', (E) has a unique solution $x(t) = x(t; t_0, x_0, x_0')$ in an appropriate interval $(t_0 - \xi, t_0 + \xi)(\xi > 0)$ with $x(t_0) = x_0$ and $x'(t_0) = x_0'$.

The zero solution of (E) is said to be *stable in the sense of Liapunov* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that every solution $x(t) = x(t; t_0, x_0, x_0')$ of (E) for which $(x_0)^2 + (x_0')^2 \leqq \delta$, satisfies the inequality $[x(t)]^2 + [x'(t)]^2 \leqq \varepsilon$ for $t \geqq t_0$ also. We say that the zero solution of (E) is *globally asymptotically stable* if every solution $x(t) = x(t; t_0, x_0, x_0')$ of (E) satisfies the relations

$$\text{(R)} \qquad \lim_{t \to \infty} x(t) = \lim_{t \to \infty} x'(t) = 0.$$

In these definitions it is understood that solutions starting near the origin exist on the whole interval $t_0 \leqq t < \infty$.

J. S. W. Wong [1] obtained a condition sufficient for the stability of the zero solution of (E). He also raised the question of finding conditions guaranteeing the global asymptotical stability of the zero solution of (E). We shall give an answer to this question.

In Sec. 1 we prove two lemmas concerning continuation, boundedness and oscillation of the solutions. In Sec. 2 we establish a *necessary* condition for the global asymptotical stability of the zero solution of (E) and a *sufficient* condition for the same property in case $b(t)$ is bounded on $[0, \infty)$. In Sec. 3 we investigate the case $\lim_{t\to\infty} b(t) = \infty$.

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1.

Let $x(t)$ be a solution of (E) and set

$$v(t) = \frac{[x'(t)]^2}{b(t)} + 2F(x(t)). \tag{1.1}$$

It is easy to see that

$$v'(t) = \frac{[x'(t)]^2}{b(t)} \left[2a(t)g(x(t), x'(t)) + \frac{b'(t)}{b(t)}\right]. \tag{1.2}$$

The non-negative function $v(t)$ will be called the *Liapunov function belonging to the solution* $x(t)$.

For the sake of brevity, we shall use the notation

$$q_l(t) = 2la(t) + \frac{b'(t)}{b(t)},$$

where l is an arbitrary real number.

Lemma 1.1. *Suppose that*

$$\int_0^\infty [q_k(t)]_- \, dt < \infty, \tag{1.3}$$

where k denotes the infimum of $g(u, v)$ on the plane (u, v). Then

a) *every solution $x(t; t_0, x_0, x_0')$ of* (E) *exists in* $[0, \infty)$;

b) *$v(t)$ is a function of bounded variation on* $[0, \infty)$, *and consequently tends to a finite limit as $t \to \infty$.*

Proof. a) Suppose that $x(t; t_0, x_0, x_0')$ is a solution of (E) and $[t_0, T)$ is the maximum interval to the right in which the solution $x(t)$ can be continued ($t_0 < T \leqq \infty$). By (1.2) we have on $[t_0, T)$

$$v'(t) \leqq v(t) \left[-2ka(t) - \frac{b'(t)}{b(t)}\right]_+ = v(t)\,[q_k(t)]_-, \tag{1.4}$$

therefore

$$\int_{t_0}^{t} \frac{v'(s)}{v(s)}\, ds \leqq \int_{t_0} [q_k(s)]_-\, ds,$$

i.e.

$$v(t) \leqq v(t_0) \exp\Big(\int_{t_0}^{T} [q_k(s)]_-\, ds\Big) = C_1, \tag{1.5}$$

thus $v(t)$ is bounded, consequently the functions $x(t)$, $x'(t)$ also are bounded on every finite subinterval of $[t_0, T)$.

Suppose now that $T<\infty$. Then $x(t)$ and $x'(t)$ are bounded on $[t_0, T)$ and by virtue of (E) $x''(t)$ is bounded too on the same interval. But $x(t)$ cannot be extended to the right of T, therefore $\lim_{t\to T-0} x(t)$ and $\lim_{t\to T-0} x'(t)$ cannot both exist, and thus either $x'(t)$ or $x''(t)$ is unbounded on $[t_0, T)$. The assumption $T<\infty$ has led to a contradiction, i.e. $x(t)$ exists in $[t_0, \infty)$.

Likewise, $x(t)$ can be continued to the left of t_0.

b) (1. 4) and (1. 5) imply $[v'(t)]_+ \leqq C_1 [q_k(t)]_-$. Since $v(t)\geqq 0$, we have

$$\int_0^\infty [v'(t)]_-\, dt \leqq v(0) + \int_0^\infty [v'(t)]_+\, dt \leqq v(0)+C_2,$$

where $C_2 = C_1 \int_0^\infty [q_k(t)]_-\, dt$; hence

$$\int_0^\infty |v'(t)|\, dt = \int_0^\infty \big([v'(t)]_+ + [v'(t)]_-\big)\, dt < \infty,$$

i.e. $v(t)$ is a function of bounded variation on $[0, \infty)$.

Corollary 1. 1. *If* (1. 3) *holds, then every solution* $x(t)$ *of* (E), *and* $x'(t)[b(t)]^{-\frac{1}{2}}$ *also, are bounded.*

Proof. In view of b), assumption (A_2) and (1. 1), the statement is obvious.

Lemma 1. 2. *Every solution* $x(t)$ *of* (E) *is either oscillatory or monotonic on an appropriate interval* $[T_0, \infty)$.

Proof. The zero solution of (E) obviously satisfies the statement of the lemma. Suppose now that $x(t)\not\equiv 0$. Then, as a consequence of the uniqueness of the zero solution of (E), $x(t)$ and $x'(t)$ have only zeros of multiplicity one and these zeros form a discrete set in every finite interval. Now to prove the lemma it is sufficient to show that between any two consecutive zeros of $x'(t)$ there is one and only one zero of $x(t)$.

Let t', t'' be two consecutive zeros of $x'(t)$. By virtue of (E) we have $x(t')x''(t')<0$, $x(t'')x''(t'')<0$, and therefore $x(t)$ has successive extremal values in t', t'', thus one of them is a maximum point, the other is a minimum point of $x(t)$. Consequently, $x''(t')$ and $x''(t'')$ are of opposite signs. Hence $x(t')$ and $x(t'')$ are also of opposite signs, and therefore $x(t)$ vanishes at some point of (t', t''). If $x(t)$ vanishes at least twice on (t', t''), then $x'(t)$ also has a zero in the same interval. This contradicts the fact that t', t'' are two consecutive zeros of $x'(t)$.

2.

Theorem 2. 1. *If*

$$\liminf_{t\to\infty} b(t) > 0 \tag{2.1}$$

and the zero solution of (E) *is globally asymptotically stable, then*

$$\int_0^\infty [q_K(t)]_+ \, dt = \infty,$$

where K is an arbitrary real number greater than $g(0, 0)$.

Proof. Let $x(t)$ be an arbitrary solution of (E). The zero solution being globally asymptotically stable, it follows from (R) and (2. 1) that $v(t)$ tends to 0 as $t\to\infty$. Since $K>g(0,0)$ and $g(u,v)$ is continuous, there exists a $\delta>0$ such that if $u^2+v^2<\delta$ then $g(u,v)<K$. Furthermore, because of (R) there exists a $T>0$ such that if $t\geqq T$ then $[x(t)]^2+[x'(t)]^2<\delta$, and hence $g\big(x(t), x'(t)\big)<K$, provided $t\geqq T$. Thus, by (1. 1), (1. 2) and assumption (A_2), we have

$$\frac{v'(t)}{v(t)} = -\frac{1}{v(t)}\frac{[x'(t)]^2}{b(t)}\left[2a(t)g\big(x(t), x'(t)\big)+\frac{b'(t)}{b(t)}\right] \geqq -\left[2Ka(t)+\frac{b'(t)}{b(t)}\right]_+$$

on $[T, \infty)$, and therefore

$$\int_T^t \frac{v'(s)}{v(s)}\, ds = \ln\frac{v(t)}{v(T)} \geqq -\int_T^t [q_K(s)]_+ \, ds$$

on the same interval. Since $v(t)$ tends to 0 as $t\to\infty$,

$$\int_0^\infty [q_K(s)]_+ \, ds \geqq \int_T^\infty [q_K(s)]_+ \, ds = \infty$$

holds, which was to be proved.

Theorem 2. 2. *Suppose that $a(t)$ and $b(t)$ are bounded on $[0, \infty)$, furthermore* (1. 3) *and* (2. 1) *are satisfied. If*

$$\int_S [q_k(t)]_+ \, dt = \infty \tag{2. 2}$$

holds on every set $S = \bigcup_{n=1}^{\infty} (a_n, b_n)$ such that

$$0 \leqq a_1, \quad a_n < b_n < a_{n+1}, \quad b_n - a_n \geqq \delta > 0 \qquad (n = 1, 2, 3, \ldots),$$

then the zero solution of (E) *is globally asymptotically stable.*

Remark 2. 1. If, say $q_k(t)$ satisfies $[q_k(t)]_+ \geqq \alpha > 0$ on $[0, \infty)$, or it is non-negative, periodic and does not vanish identically on any subinterval of $[0, \infty)$, then (2. 2) is obviously satisfied.

It is easy to prove that (2. 2) and the following statement are equivalent: for every $\delta > 0$

$$\liminf_{t \to \infty} \int_t^{t+\delta} [q_k(s)]_+ \, ds > 0$$

is valid.

Proof. Let $x(t)$ be a solution of (E). By Lemma 1. 1 $x(t)$ exists in $[0, \infty)$; $x(t)$ and $u(t) = [x'(t)]^2 [b(t)]^{-1}$ are bounded and $v(t)$ is a function of bounded variation on $[0, \infty)$.

First, we shall prove that

$$\lim_{t \to \infty} u(t) = 0. \tag{2. 3}$$

Suppose (2. 3) is false, i.e.

$$\limsup_{t \to \infty} u(t) = \lambda > 0, \tag{2. 4}$$

and consider the open unbounded set

$$H = \left\{ t : t \geqq 0, \, u(t) > \frac{\lambda}{3} \right\}. \tag{2. 5}$$

As $v(t)$ is a function of bounded variation, we have

$$\infty > \int_0^{\infty} |v'(t)| \, dt \geqq \int_H u(t) \left| 2a(t) g\big(x(t), x'(t)\big) + \frac{b'(t)}{b(t)} \right| dt \geqq \frac{\lambda}{3} \int_H [q_k(t)]_+ \, dt, \tag{2. 6}$$

and therefore, as a consequence of (2. 2), H does not contain an interval of the type (T, ∞), and hence

(2. 7) $$\liminf_{t\to\infty} u(t) \leqq \frac{\lambda}{3}.$$

Inequalities (2. 4) and (2. 7) imply that there exists a sequence of intervals $\omega_m = (t'_m, t''_m) \subset H$ $(m=1, 2, 3, \ldots)$ such that

(2. 8) $$t'_m < t''_m < t'_{m+1}, \quad u(t'_m) = u(t''_m) = \frac{\lambda}{3} \qquad (m=1, 2, 3, \ldots),$$

$\lim_{m\to\infty} t'_m = \infty$ and for every m there exists a $\tau_m \in \omega_m$ with

(2. 9) $$u(\tau_m) = \frac{2}{3}\lambda.$$

From (2. 6) and (2. 8), by assumption (2. 2), we obtain

(2. 10) $$\liminf_{m\to\infty} \operatorname{mes}(\omega_m) = 0.$$

Since $u' = v' - 2[F(x)]'$, (2. 8) and (2. 9) imply

(2. 11) $$\frac{\lambda}{3} \leqq \int_{\omega_m} |u'(t)|\,dt \leqq \int_{\omega_m} u(t) \left| 2a(t) + g(x(t), x'(t)) + \frac{b'(t)}{b(t)} \right| dt + $$
$$+2 \int_{\omega_m} |x'(t)|\,|f(x(t))|\,dt \qquad (m=1, 2, 3, \ldots);$$

moreover, in view of (2. 6) and (2. 8) we have

(2. 12) $$\lim_{m\to\infty} \int_{\omega_m} u(t) \left| 2a(t) g(x(t), x'(t)) + \frac{b'(t)}{b(t)} \right| dt = 0.$$

By virtue of assumption (2. 1), and the boundedness of $x(t)$ and $x'(t)[b(t)]^{-\frac{1}{2}}$, we get

(2. 13) $$|f(x(t))| < C_1, \quad |x'(t)| < C_2 \qquad (0 \leqq t < \infty).$$

From (2. 11) we obtain, in view of (2. 10), (2. 12) and (2. 13), the inequality

$$\frac{\lambda}{3} \leqq o(1) + 2C_1 C_2 \int_{\omega_m} ds \qquad (m \to \infty),$$

which, as a consequence of (2. 10), contradicts the fact that $\lambda > 0$; consequently (2. 3) is true. Then, in view of assumption (2. 1), it follows

(2. 14) $$\lim_{t\to\infty} x'(t) = 0.$$

It remains to verify that $x(t)$ tends to 0 as $t\to\infty$.

By Lemma 1. 1 $v(t)$ tends to a finite limit as $t\to\infty$, therefore it follows from (2. 3) that $\lim_{t\to\infty} F(x(t))$ also exists; thus taking assumption (A_2) into consideration it is easy to see that $\lim_{t\to\infty} x(t)=v$ exists too. By Lemma 1. 2 $x(t)$ is either oscillatory or monotonic for t large enough. In the first case we have obviously $v=0$; thus it is sufficient to study the second case.

Suppose that $x(t)$ is monotonic for t large enough and $v\neq 0$. Then, by virtue of (2. 1) and (A_2), we get

$$\liminf_{t\to\infty} b(t)\,|f(x(t))|>0. \tag{2.15}$$

Since $a(t)$ is bounded and $g(u, v)$ is continuous, in view of (2. 14) we have

$$\lim_{t\to\infty} a(t)g(x(t), x'(t))x'(t)=0. \tag{2.16}$$

Using (2. 15) and (2. 16) we deduce from (E) that $\liminf_{t\to\infty}|x''(t)|>0$ which contradicts the fact that $x(t)$ is bounded. Thus $v=0$, and this concludes the proof of the theorem.

Remark 2. 2. By taking $g(x, x')\equiv 1$ and $f(x)\equiv x^{2n-1}$, Theorem 2. 2 contains as a special case a sharpened form of a theorem of J. Jones (Theorem 4 of [3]).

3.

Theorem 3.1. *Suppose that*

$$\lim_{t\to\infty} b(t)=\infty, \qquad \inf_{-\infty<u,\, v<\infty} g(u, v)=k>0, \tag{3.1}$$

and for any positive number C

$$\sup_{|u|<C,\, -\infty<v<\infty} g(u, v)=K_C<\infty.$$

If

$$\liminf_{t\to\infty}\frac{q_k(t)}{[b(t)]^{\frac{1}{2}}}>0 \quad \textit{and} \quad \liminf_{t\to\infty}\frac{[b(t)]^{\frac{1}{2}}}{a(t)}>0, \tag{3.2}$$

then for every solution $x(t)$ *of* (E) *we have*

$$\lim_{t\to\infty} x(t)=\lim_{t\to\infty}\frac{x'(t)}{[b(t)]^{\frac{1}{2}}}=0.$$

Proof. This is similar to that of Theorem 2. 2.

Let $x(t)$ be a solution of (E). By virtue of (3. 2) we have $[q_k(t)]_-\equiv 0$ on an appropriate interval $[T_0, \infty)$, thus by Lemma 1. 1 $x(t)$ exists in $[0, \infty)$; furthermore

$x(t)$ and $u(t)=[x'(t)]^2[b(t)]^{-1}$ are bounded and $v(t)$ is non-negative and decreasing on $[T_0, \infty)$.

First we shall prove that

$$\lim_{t\to\infty} u(t) = 0. \tag{3. 3}$$

Suppose that (3. 3) is false, i.e. (2. 4) holds, and consider the set H defined by (2. 5). In view of (3. 2) there exists a positive number c such that if T ($T \geqq T_0$) is large enough, then

$$v(T) > \int_T^\infty |v'(t)|\,dt = \int_T^\infty u(t)\left[2a(t)g(x(t), x'(t)) + \frac{b'(t)}{b(t)}\right] dt > \tag{3. 4}$$

$$> c\frac{\lambda}{3} \int_{H\cap[T,\infty)} [b(t)]^{\frac{1}{2}}\,dt;$$

hence by (3. 1) we get that mes $(H) < \infty$. Thus, the present assumptions also imply (2. 7) and there exists a sequence of intervals $\omega_m \subset H$ ($m=1, 2, 3, \dots$) with (2. 8) and (2. 9). Combining (3. 2) and (3. 4) we obtain the estimate

$$v(T) > \int_T^\infty |v'(t)|\,dt \geqq \left(\frac{\lambda}{3}\right)^{\frac{1}{2}} \int_{H\cap[T,\infty)} |x'(t)| \frac{q_k(t)}{[b(t)]^{\frac{1}{2}}}\,dt \geqq c\left(\frac{\lambda}{3}\right)^{\frac{1}{2}} \int_{H\cap[T,\infty)} |x'(t)|\,dt,$$

from which it follows that

$$\lim_{m\to\infty} \int_{\omega_m} |x'(t)|\,dt = 0. \tag{3. 5}$$

From (2. 11) using (2. 12), (3. 5) and the boundedness of $x(t)$ we get the inequality

$$\frac{\lambda}{3} \leqq o(1) + 2C_1 \int_{\omega_m} |x'(t)|\,dt = o(1) \qquad (m\to\infty),$$

which contradicts the fact, that $\lambda > 0$. Thus, (3. 3) is true.

It remains to verify that $x(t)$ tends to 0 as $t\to\infty$.

Similarly as in the proof of Theorem 2. 2, we may restrict ourselves to the case where $x(t)$ is monotonic for t large enough. Denote by ν the limit of $x(t)$ as $t\to\infty$ (as $x(t)$ is bounded, ν is finite), and suppose $\nu \neq 0$. Then $\liminf_{t\to\infty} |x'(t)| = 0$, from which it follows that for the function $w(t) = x'(t)[x(t)]^{-1}$

$$\liminf_{t\to\infty} |w(t)| = 0 \tag{3. 6}$$

holds too. On the other hand, (E) implies that $w(t)$ satisfies the relation

$$(3.7) \qquad w'(t) = -b(t)\left[\frac{(w(t))^2}{b(t)} + \frac{a(t)}{[b(t)]^{\frac{1}{2}}} g(x(t), x'(t)) \frac{w(t)}{[b(t)]^{\frac{1}{2}}} + \frac{f(x(t))}{x(t)}\right]$$

for t large enough. In consequence of (3. 3) and $v \neq 0$, we have

$$\lim_{t\to\infty} \frac{w(t)}{[b(t)]^{\frac{1}{2}}} = 0,$$

thus (3. 1), (3. 2), (3. 7), (A_2) and assumption $v \neq 0$ imply that

$$(3.8) \qquad \lim_{t\to\infty} |w'(t)| = \infty.$$

This contradicts (3. 6). Therefore we have $v=0$.

This concludes the proof.

Corollary. *Suppose $b(t)$ is increasing on an interval $[T, \infty)$ and $\lim_{t\to\infty} b(t) = \infty$. If there exist positive constants k, K, c, C, T_1 such that*

$$(3.9) \quad k < g(u, v) < K \quad (-\infty < u, v < \infty), \qquad c \leqq \frac{[b(t)]^{\frac{1}{2}}}{a(t)} \leqq C \quad (T \leqq T_1 \leqq t < \infty),$$

then for every solution $x(t)$ of (E) *we have*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x'(t)}{[b(t)]^{\frac{1}{2}}} = 0.$$

Proof. If $t \in [T_1, \infty)$, then

$$\frac{q_k(t)}{[b(t)]^{\frac{1}{2}}} = \frac{1}{[b(t)]^{\frac{1}{2}}}\left[2ka(t) + \frac{b'(t)}{b(t)}\right] \geqq 2k \frac{a(t)}{[b(t)]^{\frac{1}{2}}},$$

thus (3. 9) implies (3. 2); therefore the assumptions of the theorem are satisfied.

References

[1] J. S. W. Wong, Boundedness theorems for certain general second order non-linear differential equations, *Monatshefte für Math.*, **71** (1967), 80—86.

[2] J. S. W. Wong, Remarks on stability conditions for the differential equation $x'' + a(t)f(x) = 0$, *J. Australian Math. Soc.*, **9** (1969), 496—502.

[3] J. Jones, On the asymptotic stability of certain second order nonlinear differential equations, *SIAM J. Appl. Math.*, **14** (1966), 16—22.

[4] J. La Salle and S. Lefschetz, *Stability by Liapunov's direct method with applications* (New York, 1961).

[5] G. Sansone, *Equazioni differenziali nel campo reale*. II, second ed. (Bologna, 1948).