On generalized absolute Cesàro summability of orthogonal series

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As usual we denote by $\sigma_n^{(\alpha)}$ the nth Cesàro means of order α of a series Σa_n. The following definition is due to FLETT [1]: A series Σa_n is said to be $|C, \alpha, \gamma|_\varkappa$ summable, where $\varkappa \geqq 1$ and $\alpha > -1$, if the series $\Sigma n^{\varkappa\gamma+\varkappa-1}|\sigma_n^{(\alpha)} - \sigma_{n-1}^{(\alpha)}|^\varkappa$ is convergent.

We prove the following theorems:

Theorem 1. *Let* $\alpha > \frac{1}{2}$, $0 \leqq \gamma < 1$, $1 \leqq \varkappa \leqq 2$. *The condition*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{2\gamma} a_n^2 \right\}^{\varkappa/2} < \infty \tag{1}$$

is necessary and suffcient that for any orthonormal system $\{\varphi_n(x)\}$ *on* $(0, 1)$ *the series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{2}$$

be summable $|C, \alpha, \gamma|_\varkappa$ *almost everywhere in* $(0, 1)$.

This theorem reduces for $\alpha > \frac{1}{2}$, $\gamma = 0$ and $\varkappa = 1$ to a theorem of LEINDLER [2] which in turn contains a theorem of TANDORI [3], case $\alpha = 1$, $\gamma = 0$, $\varkappa = 1$.

The sequence of ideas of our proof is similar to that of LEINDLER.

Theorem 2. *Let* $0 \leqq \gamma < 1$ *and* $1 \leqq \varkappa \leqq 2$. *Then the conditions*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{2\gamma} a_n^2 \log n \right\}^{\varkappa/2} < \infty \quad \left(\text{for } \alpha = \frac{1}{2} \right)$$

and

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} n^{1+2\gamma-2\alpha} a_n^2 \right\}^{\varkappa/2} < \infty \quad \left(\text{for } -1 < \alpha < \frac{1}{2} \right)$$

are sufficient that the series (2) *be summable* $|C, \alpha, \gamma|_\varkappa$ *for any orthonormal system* $\{\varphi_n(x)\}$ *in* $(0, 1)$, *almost everywhere in* $(0, 1)$.

In the special case $\gamma = 0$ $\varkappa = 1$ this theorem was proved by LEINDLER ([2], p. 253). The proof is similar to the proof of Theorem 1, so we omit it.

It is of some interest to remark the following corollary to Theorems 1 and 2 and to a theorem of FLETT (see [1], p. 359).

Corollary. *Let* $0 \leqq \gamma < 1$ *and* $\varkappa \geqq 2$. *The series* (2) *is* $|C, \alpha, \gamma|_\varkappa$ *summable for any orthonormal system* $\{\varphi_n(x)\}$ *almost everywhere in* $(0, 1)$ *in each of the following three cases:*

$$\text{(i)} \quad \alpha > 1 - \frac{1}{\varkappa} \quad \text{and} \quad \sum_{n=0}^{\infty} n^{2\gamma} a_n^2 < \infty,$$

$$\text{(ii)} \quad \alpha = 1 - \frac{1}{\varkappa} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{2\gamma} a_n^2 \log n < \infty,$$

$$\text{(iii)} \quad \alpha \geqq \beta + \frac{1}{2} - \frac{1}{\varkappa} \quad \left(-1 < \beta < \frac{1}{2}\right) \quad \text{and}$$

$$\sum_{n=0}^{\infty} n^{1+2\gamma-2\beta} a_n^2 < \infty, \quad 0 \leqq \gamma < \min(1, 1+\beta).$$

Proof of Theorem 1. Let $A_m^{(\alpha)} = \binom{m+\alpha}{m}$. Then we have:

$$0 < c_1(\alpha) \leqq \frac{A_m^{(\alpha)}}{m^\alpha} \leqq c_2(\alpha) \qquad (m > 0, \ \alpha > -1), \tag{3}$$

$$A_m^{(\alpha)} > 0 \qquad (m \geqq 0, \ \alpha > -1), \tag{4}$$

and

$$A_{m+1}^{(\alpha)} > A_m^{(\alpha)} \qquad (m \geqq 0, \ \alpha > 0), \tag{5}$$

where $c_1(\alpha)$ and $c_2(\alpha)$ are independent of m. (See ZYGMUND [4], p. 77.) We define

$$L_{n,\nu}^{(\alpha)} = \frac{A_{n+1-\nu}^{(\alpha)}}{A_{n+1}^{(\alpha)}} - \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} = \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} \cdot \frac{\nu\alpha}{(n+1-\nu)(n+1+\alpha)}.$$

From (3), (4) and (5) it easily follows that for any $n = 1, 2, \ldots$; $\nu = 0, 1, \ldots, n$; $\alpha > -1$, $\alpha \neq 0$:

$$0 < d_1(\alpha) \frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \leqq |L_{n,\nu}^{(\alpha)}| \leqq d_2(\alpha) \frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \tag{6}$$

and

$$\operatorname{sgn} L_{n,\nu}^{(\alpha)} = \operatorname{sgn} \alpha,$$

where $d_1(\alpha)$ and $d(\alpha)$ are independent of n.

First we prove the necessity of condition (1). Without loss of generality we may assume that $a_0 = a_1 = 0$ and $a_n \neq 0$ for $n \geqq 2$. We define by induction a special orthonormal system of step functions $\{\chi_n(x)\}$ $(n = 0, 1, \ldots)$ in $(0, 1)$. Let

$$\chi_n(x) = r_n(x) \qquad (n = 0, 1, 2).^{1)}$$

[1]) $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ the n-th Rademacher function.

Let $s(\geqq 1)$ be any natural number. Suppose that the step functions $\chi_n(x)$ $(n=0, 1, \ldots, 2^s)$ have been defined such that $\{\chi_n(x)\}$ $(n=0, \ldots, 2^s)$ is a H-type system i.e. $\chi_n(x)\chi_m(x)=0$ for any $x\in(0, 1)$, if $2^k<n, m\leqq 2^{k+1}$ $n\neq m$ and $k=0, 1, \ldots$ $\ldots, s-1$. Then the interval $(0, 1)$ can be dissected into subintervals J_ϱ $(1\leqq\varrho\leqq\varrho_s)$ such that on any J_ϱ every $\chi_n(x)$ $(n=0, 1, \ldots, 2^s)$ is constant. We define the following sequence:

$$\varrho_0^{(m)}=0 \quad\text{and}\quad \varrho_k^{(m)}=\frac{1}{A_m^{2/\varkappa}}\sum_{n=1}^{k} a_{2^m+n}^2 \qquad (k=1, \ldots, 2^m),$$

where $A_m=\left\{\sum_{n=2^m+1}^{2^{m+1}} a_n^2\right\}^{\varkappa/2}$ $(m=0, 1, \ldots)$. Now we dissect every interval $J_\varrho=(u_\varrho, v_\varrho)$ into 2^s intervals as follows:

$$I_k(s, J_\varrho)=(u_\varrho^{(k)}, v_\varrho^{(k)}),$$

where

$$u_\varrho^{(k)}=u_\varrho+\mu(J_\varrho)\varrho_{k-1}^{(s)} \quad\text{and}\quad v_\varrho^{(k)}=u_\varrho+\mu(J_\varrho)\varrho_k^{(s)} \qquad (k=1, \ldots, 2^s).\ ^{2)}$$

Then we define

$$\chi_{2^s+k}(x)=\frac{A_s^{1/\varkappa}}{a_{2^s+k}}\sum_{\varrho=1}^{\varrho_s} r_s(x; I_k(s, J_\varrho)).^{3)}$$

These functions $\chi_n(x)$ $(2^s\leqq n\leqq 2^{s+1})$ are step functions and

$$\int_0^1 \chi_{2^s+k}^2(x)\,dx=\frac{A_s^{2/\varkappa}}{a_{2^s+k}^2}\sum_{\varrho=1}^{\varrho_s}\int_0^1 r_s^2(x; I_k(s, J_\varrho))\,dx=$$

$$=\frac{A_s^{2/\varkappa}}{a_{2^s+k}^2}\sum_{\varrho=1}^{\varrho_s}\frac{a_{2^s+k}^2}{A_s^{2/\varkappa}}\mu(J_\varrho)=\sum_{\varrho=1}^{\varrho_s}\mu(J_\varrho)=1.$$

From the definition it is clear that the functions $\chi_n(x)$ $(n=0, 1, \ldots, 2^{s+1})$ give rise to an orthonormal system on $(0, 1)$ and for every $x\in(0, 1)$ we have

$$\chi_n(x)\chi_m(x)=0 \qquad (2^l<n, m\leqq 2^{l+1};\ 0\leqq l\leqq s).$$

Hence, by induction, we get an infinite H-type system.

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[2]) μ denotes Lebesgue measure.

[3]) If $I(u, v)$ is a finite interval and $h(x)$ is a function defined on $(0, 1)$, then

$$h(x; I)=\begin{cases} h\left(\dfrac{x-u}{v-u}\right), & \text{if } u<x<v\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $\int_u^v h(x; I)\,dx=\mu(I)\int_0^1 h(x)\,dx$.

We consider the series

$$\sum_{n=0}^{\infty} a_n \chi_n(x). \tag{7}$$

Denote $\bar{\sigma}_n^{(\alpha)}$ the n-th (C, α) means of series (7). Let us assume that the series (2) for any orthonormal system is $|C, \alpha, \gamma|_\varkappa$ summable almost everywhere in (0, 1). Then we have

$$\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha)}(x)|^\varkappa < \infty$$

almost everywhere in (0, 1).

Let $\varepsilon = \min\{1; 2^{-(7+3\varkappa+3\varkappa\alpha)} d_1^\varkappa(\alpha) d_2^\varkappa(\alpha)\}$, where $d_1(\alpha)$ and $d_2(\alpha)$ are the same constants as in (6). By the Egorov theorem there exists a measurable set E with $\mu(E) \geqq 1-\varepsilon$ and a positive constant K such that for every $x \in E$

$$\sum_{n=1}^{\infty} n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha)}(x)|^\varkappa < K.$$

Hence

$$\sum_{n=2}^{\infty} \int_E n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha-1)}(x)|^\varkappa \leqq K\mu(E).$$

Let m and n be integers such that $2^m < n \leqq 2^{m+1}$. Then we put

$$R_l(x; m, n) = \sum_{\nu=2^l+1}^{2^{l+1}} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x) \qquad (l=0, 1, \ldots, m-1),$$

$$R_m(x; m, n) = \sum_{\nu=2^m+1}^{n} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x), \quad R_{m+1}(x; m, n) = \frac{1}{A_{n+1}^{(\alpha)}} a_{n+1} \chi_{n+1}(x).$$

These functions $R_l(x; m, n)$ $(l = 0, 1, \ldots, m+1)$ satisfy the conditions of the following

Lemma. (LEINDLER [2]) *Let $\{R_n(x)\}$ $(n=1, 2, \ldots)$ be a system of step functions defined on (0, 1). Denote $J_s(n)$ $(n=1, 2, \ldots;\ s=1, 2, \ldots, s_n)$ the intervals on which $R_n(x)$ is constant. If for every $m > n$*

$$\int_{J_s(n)} \operatorname{sign} R_m(x)\, dx = 0 \qquad (s=1, \ldots, s_n),$$

then for any sequence of numbers $d_1, \ldots, d_N$ there exists a set E_k of subintervals such that for any $x \in E_k$

$$\left|\sum_{l=1}^{N} d_l R_l(x)\right| \geqq |d_{N-k} R_{N-k}(x)| \qquad (k=0, 1, \ldots, N-1)$$

and

$$\mu(E_k \cap J_s(N-k-1)) = \frac{\mu(J_s(N-k-1))}{2^{k+1}}$$

$$(k=0, 1, \ldots, N-1;\ s=1, 2, \ldots, s_{N-k-1};\ J_1(0)=(0, 1)).$$

We use this lemma in case $N = m+1$ and $k=3$. The suitable set E_k will be denoted by $E_3(m, n)$. Then we have:

$$(8) \qquad \sum_{n=2^3+1}^{\infty} \int_E n^{\varkappa\gamma+\varkappa-1} |\bar{\sigma}_n^{(\alpha)}(x) - \bar{\sigma}_{n-1}^{(\alpha)}(x)|^{\varkappa}\, dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \int_E \left| \sum_{\nu=0}^{n} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x) + \frac{1}{A_{n+1}^{(\alpha)}} a_{n+1}\chi_{n+1}(x) \right|^{\varkappa} dx =$$

$$= \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \int_E \left| \sum_{l=0}^{m+1} R_l(x; m, n) \right|^{\varkappa} dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} \int_{E \cap E_3(m, n)} |R_{m-2}(x; m, n)|^{\varkappa}\, dx \geqq$$

$$\geqq \sum_{m=3}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \left(\int_{E_3(m, n)} - \int_{E_3(m, n) - E_3(m, n) \cap E} \right) |R_{m-2}(x; m, n)|^{\varkappa}\, dx = S.$$

By the lemma we have:

$$\int_{E_3(m, n)} |R_{m-2}(x; m, n)|^{\varkappa}\, dx = \int_{E_3(m, n)} \left| \sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)} a_\nu \chi_\nu(x) \right|^{\varkappa} dx =$$

$$= \int_{E_3(m, n)} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^{\varkappa} a_\nu^{\varkappa} |\chi_\nu(x)|^{\varkappa}\, dx \geqq$$

$$\geqq \sum_{k=1}^{2^{m-2}} (L_{n,2^{m-2}+k}^{(\alpha)})^{\varkappa} a_{2^{m-2}+k}^{\varkappa} \sum_{\varrho=1}^{\varrho_{m-2}} \int_{E_3(m, n) \cap I_k(m-2, J_\varrho)} \frac{A_{m-2}}{a_{2^{m-2}+k}^{\varkappa}}\, dx \geqq$$

$$\geqq \sum_{k=1}^{2^{m-2}} (L_{n,2^{m-2}+k}^{(\alpha)})^{\varkappa} A_{m-2} \sum_{\varrho-1}^{\varrho_{m-2}} \frac{\mu(I_k(m-2, J_\varrho))}{2^4} =$$

$$= \frac{1}{2^4} \sum_{k=1}^{2^{m-2}} (L_{n,2^{m-2}+k}^{(\alpha)})^{\varkappa} A_{m-2} \sum_{\varrho-1}^{\varrho_{m-2}} \frac{a_{2^{m-2}+k}^2}{A_{m-2}^{2/\varkappa}} \mu(J_\varrho) =$$

$$= \frac{1}{2^4} \sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^{\varkappa} A_{m-2}^{1-2/\varkappa} a_\nu^2.$$

In order to estimate the second integral in (8) we apply the Hölder inequality:

$$\int\limits_{E_3(m,n)-E_3(m,n)\cap E} |R_{m-2}(x;m,n)|^{\varkappa}\,dx \leqq \varepsilon^{\frac{2-\varkappa}{2}}\left(\int_0^1\left|\sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)}a_\nu\chi_\nu(x)\right|^2 dx\right)^{\varkappa/2} =$$

$$= \varepsilon^{\frac{2-\varkappa}{2}}\left(\sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^2 a_\nu^2\right)^{\varkappa/2}.$$

Thus, by a standard computation, we obtain that

$$S \geqq \sum_{m=3}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1}\cdot$$

$$\cdot\left\{2^{-4}\sum_{\nu=2^{m-2}+1}^{2^{m-1}} (L_{n,\nu}^{(\alpha)})^{\varkappa} A_{m-2}^{1-2/\varkappa} a_\nu^2 - \varepsilon^{\frac{2-\varkappa}{2}}\left(\sum_{\nu=2^{m-2}+1}^{2^{m-1}} L_{n,\nu}^{(\alpha)})^2 a_\nu^2\right)^{\varkappa/2}\right\} \geqq$$

$$\geqq \sum_{m=3}^{\infty} 2^{-4} d_1^{\varkappa}(\alpha) A_{m-2}^{1-2/\varkappa}\sum_{\nu=2^{m-2}+1}^{2^{m-1}} n^{\varkappa\gamma+\varkappa-1}\left(\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}}\right)^{\varkappa} -$$

$$-\sum_{m=3}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1} d_{\varkappa}^2(\alpha)\,\varepsilon^{\frac{2-\varkappa}{2}}\left(\sum_{\nu=2^{m-2}+1}^{2^{m-1}}\left(\frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}}\right)^2 a_\nu^2\right)^{\varkappa/2} \geqq$$

$$\geqq d_1^{\varkappa}(\alpha) 2^{-(6+\varkappa+2\varkappa\alpha-\varkappa\gamma)}\sum_{m=1}^{\infty}\left\{\sum_{\nu=2^m+1}^{2^{m+1}} \nu^{2\gamma}a_\nu^2\right\}^{\varkappa/2},$$

i.e. the necessity of condition (1) is proved.

Next we prove that condition (1) is also sufficient. We suppose, as we may do without loss of generality, that $a_0 = a_1 = 0$. Applying (3), (6), and the Hölder inequality we have:

$$\sum_{n=3}^{\infty} n^{\varkappa\gamma+\varkappa-1}\int_0^1 |\sigma_n^{(\alpha)}(x) - \sigma_{n-1}^{(\alpha)}(x)|^{\varkappa}\,dx \leqq$$

$$\leqq O(1)\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} n^{\varkappa\gamma+\varkappa-1}\left(\int_0^1 |\sigma_{n+1}^{(\alpha)}(x) - \sigma_n^{(\alpha)}(x)|^2\,dx\right)^{\varkappa/2} \leqq$$

$$\leqq O(1)\sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})}\left(\sum_{n=2^m+1}^{2^{m+1}}\left(\sum_{\nu=0}^{n}(L_{n,\nu}^{(\alpha)})^2 a_\nu^2 + \frac{1}{(A_{n+1}^{(\alpha)})^2}a_{n+1}^2\right)\right)^{\varkappa/2} =$$

$$= O(1)\sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})}\left(\sum_{n=2^m+1}^{2^{m+1}}\sum_{\nu=0}^{n}\frac{(n+1-\nu)^{2\alpha-2}}{n^{2\alpha+2}}\nu^2 a_\nu^2\right)^{\varkappa/2} +$$

$$+ O(1)\sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})}\left(\sum_{n=2^m+2}^{2^{m+1}+1}\frac{a_n^2}{(A_n^{(\alpha)})^2}\right)^{\varkappa/2} = O(1)\left(\sum\nolimits_1 + \sum\nolimits_2\right).$$

A standard computation shows that

$$\sum\nolimits_1 \leqq O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+1}^{2^{m+1}} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{\min(2^{l+1},n)} \frac{(n+1-\nu)^{2\alpha-2}\nu^2 a_\nu^2}{n^{2\alpha+2}} \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{m(2\gamma-1-2\alpha)} \sum_{n=2^m+1}^{2^{m+1}} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{\min(2^{l+1},n)} (n+1-\nu)^{2\alpha-2}\nu^2 a_\nu^2 \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{m(2\gamma-1-2\alpha)} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{2^{l+1}} \nu^2 a_\nu^2 \sum_{n=\max(2^m+1,\nu)}^{2^{m+1}} (n+1-\nu)^{2\alpha-2} \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{2m(\gamma-1)} \sum_{l=0}^{m} \sum_{\nu=2^l+1}^{2^{l+1}} \nu^2 a_\nu^2 \right)^{\varkappa/2} \leqq$$

$$\leqq O(1) \sum_{l=0}^{\infty} 2^{\varkappa l} \left(\sum_{\nu=2^l+1}^{2^{l+1}} a_\nu^2 \right)^{\varkappa/2} \sum_{m=l}^{\infty} 2^{m\varkappa(\gamma-1)} = O(1) \sum_{l=0}^{\infty} \left(\sum_{\nu=2^l+1}^{2^{l+1}} \nu^{2\gamma} a_\nu^2 \right)^{\varkappa/2}$$

and

$$\sum\nolimits_2 \leqq O(1) \sum_{m=0}^{\infty} 2^{\varkappa m(\gamma+\frac{1}{2})} \left(\sum_{n=2^m+1}^{2^{m+1}} \frac{a_n^2}{n^{2\alpha}} \right)^{\varkappa/2} + O(1) \sum_{m=0}^{\infty} 2^{\varkappa\gamma m+\varkappa m(\frac{1}{2}-\alpha)} a_{2^m+1}^{\varkappa} \leqq$$

$$\leqq O(1) \sum_{m=0}^{\infty} \left(2^{2\gamma m} \sum_{n=2^m+1}^{2^{m+1}} a_n^2 \right)^{\varkappa/2} \leqq O(1) \sum_{m=0}^{\infty} \left(\sum_{n=2^m+1}^{2^{m+1}} n^{2\gamma} a_n^2 \right)^{\varkappa/2}.$$

By the Beppo Levi theorem our proof is complete.

References

[1] T. M. FLETT, Some more theorems concerning the absolute summability of Fourier series and power series, *Proc. London Math. Soc.*, (3) **8** (1958), 357—387.

[2] L. LEINDLER, Über die absolute Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **22** (1961), 243—268.

[3] K. TANDORI, Über die orthogonalen Functionen. IX. Absolute Summation, *Acta Sci. Math.*, **21** (1960), 292—299.

[4] A. Z. ZYGMUND, *Trigonometric series*. I (Cambridge, 1959).

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