On the divergence of rearranged Fourier series of square integrable functions

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Introduction

K. TANDORI [3] gave an elementary proof to the statement of A. N. KOLMOGOROFF [1] that there exists a square integrable function whose Fourier series can be rearranged so as to diverge almost everywhere. He [4] also proved the following theorem:

Theorem A. *If* $\{\varrho(n)\}$ *is a sequence of positive numbers with*

(1) $$\varrho(n) = o(\sqrt{\log\log n}),$$

then there exists a sequence $\{c_n\}$ *with* $\Sigma c_n^2 \varrho^2(n) < \infty$ *such that the Walsh series* $\Sigma c_n w_n(x)$ *diverges almost everywhere in* $(0, 1)$ *in a certain rearrangement of its terms.*

Afterwards F. MÓRICZ [2] showed a generalization of [3] which can be considered as a trigonometric series analogue of Theorem A. That is:

Theorem B. *Suppose* (1). *Then there exists a square integrable function whose Fourier series* $\Sigma(a_n \cos nx + b_n \sin nx)$ *is such that*

(2) $$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\varrho^2(n) < \infty$$

and which can be rearranged into an everywhere divergent series.

In the present paper we will sharpen Theorem B by refining the method of ist proof.

Theorem. *If* $\{\varrho(n)\}$ *is a sequence of positive numbers with*

(3) $$\varrho(n) = o(\sqrt[4]{\log n}),$$

then there exists a square integrable function whose Fourier series fulfils (2) *and which can be rearranged into an everywhere divergent series.*

Corollary. *Suppose* (3), *then there exists a square integrable function whose Fourier series can be rearranged in such a way that the partial sums* $\sigma_N(x)$ *of the rearranged series satisfy*

$$\limsup_{N\to\infty} \frac{|\sigma_N(x)|}{\varrho(N)} > 0 \text{ everywhere.}$$

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§ 1. Lemmas

Consider a set $E = \bigcup_{i=1}^{m} J_i$ satisfying $\bar{J}_i \cap \bar{J}_j = \emptyset \ (i \neq j)$ [1]) and $\max_i |J_i| > 0$.[2]) If each J_i is an interval, then E is said to be *simple*, and we write $E \in \mathcal{S}$. More generally, if each J_i is either an interval or a point, then E is said to be *generalized simple*, and we write $E \in \mathcal{S}^*$. Suppose $E \in \mathcal{S}^*$, then for $0 < \varepsilon < \max_i |J_i|/2$, we set

$$E^{(\varepsilon)} = \bigcup_{\beta_i - \alpha_i > 2\varepsilon} [\alpha_i + \varepsilon, \beta_i - \varepsilon]$$

where α_i and β_i denote the left and right end points of J_i respectively. It is obvious that $E^{(\varepsilon)} \in \mathcal{S}$.

For a function $a_\nu \cos \nu x + b_\nu \sin \nu x \ (\not\equiv 0)$ we call ν its frequency. Two trigonometric polynomials are called disjoint if they have no terms of the same frequency.

$C_1, C_2, \ldots$ denote positive absolute constants which will be common in several lemmas.

Lemma 1. *Let* $E = \bigcup_i J_i \in \mathcal{S}^*$ *be a subset of* $[-\pi/12, \pi/12]$, $0 < \varepsilon < \max_i |J_i|/2$ and $0 < \eta \leqq 1$ *real numbers, and* n *a natural number such that* $n > C_1/\varepsilon\eta - 1$ $(C_1 = \pi)$. *Then there exists a non-negative trigonometric polynomial* $P(x)$ *with frequences* 6ν $(\nu = 0, 1, \ldots, n)$ *such that*

$$P(x) \geqq 1 \quad \textit{for} \quad x \in E^{(\varepsilon)}, \tag{7}$$

$$P(x) \leqq \eta \quad \textit{for} \quad x \in \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - E \tag{8}$$

and

$$\int_{-\pi}^{\pi} P^2(x)\,dx \leqq C_2 |E| \qquad \left(C_2 = \frac{27}{4}\pi^4\right). \tag{9}$$

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[1]) J_i denotes the closure of J_i.

[2]) $|J_i|$ denotes the Lebesgue measure of J_i.

We can verify Lemma 1 with the aid of the proof of the similar lemma in [2], so we omit its proof.

Lemma 2. *Take the same assumptions and notations as in Lemma 1, an let $N\,(\geqq 12n+6)$ be a natural number divisible by 6. Furthermore set*

$$
\begin{aligned}
&Q_1(x) = (\cos Nx)\,P(x),\\
(10)\qquad &Q_2(x) = -C_3(\cos 3x)(\cos Nx)\,P(x) \qquad (C_3 = 2\sqrt{2}),\\
&Q_3(x) = -C_4(\cos 2Nx)\,P(x) \qquad (C_4 = 3+4\sqrt{2}).
\end{aligned}
$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequences 3ν having the following properties:

(11) $$N-6n-3 \leqq 3\nu \leqq N+6n+3 \quad \text{or} \quad 2N-6n \leqq 3\nu \leqq 2N+6n;$$

(12) $$|Q_1(x)+Q_2(x)+Q_3(x)| \leqq C_5\eta \quad \text{for} \quad x\in\left[-\frac{\pi}{12},\,\frac{\pi}{12}\right]-E$$

$$(C_5 = 1+C_3+C_4);$$

(13) $$\int_{-\pi}^{\pi} |Q_1(x)+Q_2(x)+Q_3(x)|^2\,dx \leqq C_6|E| \qquad (C_6 = C_2(1+C_3^2+C_4^2));$$

there exists a decomposition $E^{(\varepsilon)} = E_1+E_2+E_3$ such that

(14) $$\sum_{k=1}^{l} Q_k(x) \geqq \frac{1}{2} \quad \text{for} \quad x\in E_l \qquad (l=1,2,3).$$

In addition if

(15) $$\frac{2\pi}{N} \leqq \max_i |J_i| - 2\varepsilon$$

is satisfied, then each E_l contains an interval whose length is not smaller than $\pi/3N$ and $E_l\in\mathcal{S}^*$ $(l=1,2,3)$.

Proof. It is easy to see that the frequencies of the terms of $Q_1(x)$ and $Q_3(x)$ are divisible by 6, and those of $Q_2(x)$ divisible by 3 but not by 6. Moreover the frequencies 3ν of the terms of $Q_1(x)$ and $Q_2(x)$ satisfy the former inequalities of (11), and those of $Q_3(x)$ only the latter ones. (12) and (13) are shown by simple calculations using (8) and (9), respectively. And in virtue of (7), the following estimates hold:

$$Q_1(x) = (\cos Nx)\,P(x) \geqq \frac{1}{2}\cdot 1 = \frac{1}{2}$$

for

$$x \in E_1 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[\frac{1}{N}\left(2k\pi - \frac{\pi}{3}\right), \frac{1}{N}\left(2k\pi + \frac{\pi}{3}\right)\right];$$

$$Q_1(x) + Q_2(x) = (C_3 \cos 3x - 1)(-\cos Nx) P(x) \geqq \left(\frac{C_3}{\sqrt{2}} - 1\right) \cdot \frac{1}{2} \cdot 1 = \frac{1}{2}$$

for

$$x \in E_2 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[\frac{1}{N}\left(2k\pi + \frac{2}{3}\pi\right), \frac{1}{N}\left(2k\pi + \frac{4}{3}\pi\right)\right];$$

$$Q_1(x) + Q_2(x) + Q_3(x) \geqq Q_3(x) - |Q_2(x)| - |Q_1(x)| > \frac{C_4}{2} - C_3 - 1 = \frac{1}{2}$$

for

$$x \in E_3 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left(\frac{1}{N}\left(k\pi + \frac{\pi}{3}\right), \frac{1}{N}\left(k\pi + \frac{2\pi}{3}\right)\right).$$

Now let us set $|J_{i_0}| = \max_i |J_i|$, and assume (15). Then in virtue of the definition of E_l $(l=1, 2, 3)$, each $E_l \cap J_{i_0}^{(\varepsilon)}$ contains an interval whose length is not smaller than $\pi/3N$. This completes the proof of Lemma 2.

Lemma 2′. *Let $P(x)$ be a trigonometric polynomial with frequencies 3ν $(\nu \leqq n)$, and $N(\geqq 6n+3)$ a natural number divisible by 3. Furthermore set*

$$(10') \qquad \begin{aligned} Q_1(x) &= (\cos Nx) P(x), \\ Q_2(x) &= -C_3 (\cos x)(\cos Nx) P(x), \\ Q_3(x) &= -C_4 (\cos 2Nx) P(x). \end{aligned}$$

Then $Q_1(x)$, $Q_2(x)$ and $Q_3(x)$ are mutually disjoint trigonometric polynomials with frequencies ν having the following properties:

$$(11') \qquad N - 3n - 1 \leqq \nu \leqq 2N + 3n;$$

$$(13') \qquad \int_{-\pi}^{\pi} |Q_1(x) + Q_2(x) + Q_3(x)|^2 \, dx \leqq C_6 \int_{-\pi}^{\pi} P^2(x) \, dx;$$

for every set $E(\subset [-\pi/4, \pi/4])$ on which $P(x)$ is positive, there exists a decomposition $E = E_1 + E_2 + E_3$ such that

$$(14') \qquad \sum_{k=1}^{l} Q_k(x) \geqq \frac{P(x)}{2} \quad \textit{for} \quad x \in E_l \qquad (l=1, 2, 3)$$

and

$$(14'') \qquad Q_l(x) \geqq \frac{P(x)}{2} \quad \textit{for} \quad x \in E_l \qquad (l=1, 2, 3).$$

The proof of Lemma 2′ is quite in an analogy to that of Lemma 2.

Lemma 3. *If* $0 < \varepsilon < \pi/6$, *then there exist mutually disjoint trigonometric polynomials* $R_k^{(i)}(x)$ *and generalized simple sets*

$$E_k^{(i)} \subset [-\pi/12, \pi/12] \qquad (k=1, 2, \dots, 3^i; \quad i=0, 1, \dots)$$

with the following properties:

(i) *the frequencies occurring in* $R_k^{(i)}(x)$ $(k=1, 2, \dots, 3^i)$ *are divisible by* 3 *and smaller than* $6f_i(\varepsilon)$ *where*

$$f_i(\varepsilon) = \left(\frac{C_7}{\varepsilon}\right)^i 18^{\frac{i(i-1)}{2}} \qquad (C_7 = [128 C_1 C_5]+1);\ ^{3)}$$

(ii) $$\int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^i} R_k^{(i)}(x)\right)^2 dx \leqq C_8 \qquad \left(C_8 = C_6 \cdot \frac{\pi}{6}\right);$$

(iii) *the sets* $E_k^{(i)}$ $(k=1, 2, \dots, 3^i)$ *corresponding to the same value of* i *are disjoint;*

(iv) *set*

$$E_k^{(i)} = \bigcup_{j=1}^{g_k^{(i)}} J_j \quad \textit{and} \quad v_i(\varepsilon) = \sum_{k=1}^{3^i} g_k^{(i)},$$

then $v_i(\varepsilon) \leqq f_i(\varepsilon)$;

(v) *set*

$$F_i = \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - \bigcup_{k=1}^{3^i} E_k^{(i)},$$

then $|F_i| \leqq \varepsilon(1-1/2^i)$;

(vi) *the trigonometric polynomials* $R_k^{(j)}(x)$ $(k=1, \dots, 3^j;\ j=0, \dots, i)$ *can be arranged into a sequence*

$$(16) \qquad U_1^{(i)}(x), U_2^{(i)}(x), \dots, U_{h(i)}^{(i)}(x) \qquad (h(i) = (3^{i+1}-1)/2),$$

such that

$$(17) \qquad \sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) \geqq \frac{i+1}{4} \quad \textit{for} \quad x \in E_k^{(i)}$$

with $m_k^{(i)}$ *not depending on the particular point* $x \in E_k^{(i)}$ $(k=1, 2, \dots, 3^i)$.

Proof. Define $R_1^{(0)}(x)=1$ and $E_1^{(0)} = [-\pi/12, \pi/12]$, then these satisfy (i)—(vi) trivially. Setting $\varkappa_i = 1/2^{i+2} f_i(\varepsilon)$ $(i \geqq 0)$, $\eta_0 = 1$ and $\eta_i = 1/C_5 4(3^i-1)$ $(i \geqq 1)$, we take natural numbers

$$n_i = \left[\frac{C_1}{\varkappa_i \varepsilon \eta_i}\right] \qquad (i=0, 1, \dots)$$

and

$$N_k^{(i)} = (2k-1)(12 n_i + 6) \qquad (k=1, 2, \dots, 3^i; \quad i=0, 1, \dots).$$

[3]) The integer part of a real number α is denoted by $[\alpha]$.

We have the following estimates:

$$(18)\quad N_1^{(i)}-6n_i-3=6n_i+3>6\left(\frac{C_1}{\varkappa_i\cdot\frac{\pi}{6}\cdot 1}-1\right)+3=36\cdot 2^{i+2}f_i(\varepsilon)-3>72f_i(\varepsilon)\qquad (i\geqq 0);$$

$$(19)\quad 2N_{3^i}^{(i)}+6n_i=\{24(2\cdot 3^i-1)+6\}n_i+12(2\cdot 3^i-1)\leqq$$

$$\leqq(48\cdot 3^i-18)\frac{C_1}{\varepsilon}2^{i+2}f_i(\varepsilon)C_5 4\cdot 3^i+24\cdot 3^i-12=$$

$$=6\cdot 18^i\frac{128C_1C_5}{\varepsilon}f_i(\varepsilon)-6^i\frac{288C_1C_5}{\varepsilon}f_i(\varepsilon)+24\cdot 3^i-12<$$

$$<6\cdot 18^i\left(\frac{C_7}{\varepsilon}\right)f_i(\varepsilon)=6\cdot f_{i+1}(\varepsilon)\qquad (i\geqq 0);$$

$$(20)\qquad 2\varkappa_i\varepsilon=\frac{2\varepsilon}{2^{i+2}f_i(\varepsilon)}<\begin{cases}\dfrac{\pi}{12} & (i=0),\\[2mm] \dfrac{\pi}{24f_i(\varepsilon)}<\dfrac{\pi}{8N_{3^{i-1}}^{(i-1)}} & (i\geqq 1).\end{cases}$$

Applying Lemma 2 to ($E_1^{(0)}$, $\varkappa_0\varepsilon$, η_0, n_0 and $N_1^{(0)}$), we get the mutually disjoint trigonometric polynomials $Q_l(x)$ $(l=1, 2, 3)$ and the decomposition $E_1^{(0)(\varkappa_0\varepsilon)}=$ $=E_1+E_2+E_3$. Define $R_k^{(1)}(x)=Q_k(x)$ and $E_k^{(1)}=E_k$ $(k=1, 2, 3)$. Then we can easily check that (i)—(vi) hold for $i=1$. For example as to (iv),

$$v_1(\varepsilon)\leqq v_0(\varepsilon)+2\left(\left[\frac{\frac{\pi}{12}-\frac{\varepsilon}{4}}{\frac{\pi}{2N_1^{(0)}}}\right]+1\right)\leqq$$

$$\leqq 3+N_1^{(0)}\left(\frac{1}{3}-\frac{\varepsilon}{\pi}\right)<3+3f_1(\varepsilon)\left(\frac{1}{3}-\frac{\varepsilon}{\pi}\right)\leqq f_1(\varepsilon)+3-3\cdot 128C_5<f_1(\varepsilon);$$

and as to (vi), we set $U_1^{(1)}(x)=R_1^{(0)}(x)$, $U_2^{(1)}(x)=R_1^{(1)}(x)$, $U_3^{(1)}(x)=R_2^{(1)}(x)$, $U_4^{(1)}(x)=$ $=R_3^{(1)}(x)$ and $m_k^{(1)}=1+k$ $(k=1, 2, 3)$. Furthermore, since $|E_1^{(0)}|-2\varkappa_0\varepsilon>\pi/6-$ $-\pi/12>2\pi/N_1^{(0)}$, we see that each $E_k^{(1)}$ $(k=1, 2, 3)$ contains an interval whose length is not smaller than $\pi/2N_1^{(0)}$ and that $E_k^{(1)}\in\mathscr{S}^*$.

Now we suppose that $R_k^{(j)}(x)$ and $E_k^{(j)}$ $(k=1, \dots, 3^j;\ j=0, \dots, i)$ are already defined and satisfy (i)—(vi), and that

$$(21)\qquad \max_{1\leqq j\leqq g_k^{(i)}}|J_j|\geqq\frac{\pi}{3N_{3^{i-1}}^{(i-1)}}$$

for each $E_k^{(i)}$ $(k=1, 2, \dots, 3^i)$. Then by (20) and (21).

$$2\varkappa_i \varepsilon < \max_{1 \le j \le g_k^{(i)}} |J_j|$$

holds for each $E_k^{(i)}$ $(k=1, 2, \dots, 3^i)$. By the application of Lemma 2 to each $(E_k^{(i)}, \varkappa_i\varepsilon, \eta_i, n_i$ and $N_k^{(i)})$ $(k=1, 2, \dots, 3^i)$, we define the mutually disjoint trigonometric polynomials

$$(22)\qquad \begin{aligned} R_{3k-2}^{(i+1)}(x) &= (\cos N_k^{(i)}x)P_k^{(i)}(x),\\ R_{3k-1}^{(i+1)}(x) &= -C_3(\cos 3x)(\cos N_k^{(i)}x)P_k^{(i)}(x),\\ R_{3k}^{(i+1)}(x) &= -C_4(\cos 2N_k^{(i)}x)P_k^{(i)}(x)\end{aligned}$$

and the decompositions

$$E_k^{(i)(\varkappa_i\varepsilon)} = E_{3k-2}^{(i+1)} + E_{3k-1}^{(i+1)} + E_{3k}^{(i+1)} \qquad (k=1, 2, \dots, 3^i).$$

In virtue of (11), the frequencies of (22) belong to $A_k \cup B_k$ where

$$A_k = [(2k+1)N_1^{(i)} - 6n_i - 3, (2k+1)N_1^{(i)} + 6n_i + 3],$$

$$B_k = [2(2k+1)N_1^{(i)} - 6n_i, 2(2k+1)N_1^{(i)} + 6n_i].$$

It is obvious that $A_k \cap A_{k'} = \emptyset$ and $B_k \cap B_{k'} = \emptyset$ for $k \ne k'$. Moreover we have $A_k \cap B_{k'} = \emptyset$ $(k \ne k')$ since, though $|A_k|/2 + |B|_{k'}/2 = (6n_i+3) + 6n_i < N_1^{(i)}$ holds, the distance of the middle points of A_k and $B_{k'}$ is not smaller than $N_1^{(i)}$. Thus the trigonometric polynomials $R_k^{(i+1)}(x)$ $(k=1, 2, \dots, 3^{i+1})$ are mutually disjoint. And we are going to show (i)—(vi) and (21) replacing i with $i+1$.

By (18) the frequencies occurring in $R_1^{(i+1)}(x)$ are larger than those of $R_k^{(i)}(x)$ $(k=1, 2, \dots, 3^i)$, and by (19) the property (i) is verified. As to (ii), we have

$$\int_{-\pi}^{\pi} \left(\sum_{k=1}^{3^{i+1}} R_k^{(i+1)}(x)\right)^2 dx = \sum_{k=1}^{3^i} \int_{-\pi}^{\pi} (R_{3k-2}^{(i+1)}(x) + R_{3k-1}^{(i+1)}(x) + R_{3k}^{(i+1)}(x))^2\, dx =$$

$$= \sum_{k=1}^{3^i} C_6 |E_k^{(i)}| \le C_6 \cdot \frac{\pi}{6} = C_8;$$

and as to (iii), it is obvious. As to (iv), setting

$$E_k^{(i)(\varkappa_i\varepsilon)} = \bigcup_l I_l^{(i,k)} \qquad (k=1, 2, \dots, 3^i),$$

we have

$$v_{i+1}(\varepsilon) \le v_i(\varepsilon) + \sum_{k=1}^{3^i} \sum_l 2\left(\left[\frac{|I_l^{(i,k)}|}{\frac{\pi}{N_k^{(i)}}}\right] + 1\right)$$

$$\le v_i(\varepsilon) + \frac{2N_{3^i}^{(i)}}{\pi} \sum_{k=1}^{3^i} |E_k^{(i)(\varkappa_i\varepsilon)}| + 2v_i(\varepsilon) \le 3f_i(\varepsilon) + \frac{6f_{i+1}(\varepsilon)}{\pi}\left(\frac{\pi}{6} - \frac{\varepsilon}{2}\right) < f_{i+1}(\varepsilon);$$

and as to (v),

$$|F_{i+1}| = \left|F_i \cup \left\{\bigcup_{k=1}^{3^i} (E_k^{(i)} - E_k^{(i)(\varkappa_i \varepsilon)})\right\}\right| \leqq |F_i| + 2\varkappa_i \varepsilon \cdot v_i(\varepsilon) \leqq$$

$$\leqq \varepsilon\left(1 - \frac{1}{2^i}\right) + \frac{2\varepsilon}{2^{i+2} f_i(\varepsilon)} f_i(\varepsilon) = \varepsilon\left(1 - \frac{1}{2^{i+1}}\right).$$

As to (vi), we define the sequence

$$U_1^{(i+1)}(x), U_2^{(i+1)}(x), \dots, U_{h(i+1)}^{(i+1)}(x)$$

by inserting $R_{3k-2}^{(i+1)}(x)$, $R_{3k-1}^{(i+1)}(x)$, $R_{3k}^{(i+1)}(x)$ after $R_k^{(i)}(x)$ $(k=1, 2, \dots, 3^i)$ in (16), and define $m_k^{(i+1)}$ $(k=1, 2, \dots, 3^{i+1})$ by

$$U_{m_k^{(i+1)}}^{(i+1)}(x) = R_k^{(i+1)}(x) \qquad (k=1, 2, \dots, 3^{i+1}).$$

Then if $x \in E_{3k-3+l}^{(i+1)}$, $1 \leqq k \leqq 3^i$ and $1 \leqq l \leqq 3$, we obtain

$$\sum_{j=1}^{m_{3k-3+l}^{(i+1)}} U_j^{(i+1)}(x) = \sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) + \sum_{j=1}^{3k-3+l} R_j^{(i+1)}(x) \geqq$$

$$\geqq \sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) + \sum_{j=1}^{l} R_{3k-3+j}^{(i+1)}(x) - \sum_{j=1}^{3^i}{}' \, |R_{3j-2}^{(i+1)}(x) + R_{3j-1}^{(i+1)}(x) + R_{3j}^{(i+1)}(x)| \geqq$$

$$\geqq \frac{i+1}{4} + \frac{1}{2} - (3^i - 1) C_5 \eta_i = \frac{(i+1)+1}{4}.$$

Finally by

$$\frac{2\pi}{N_1^{(i)}} < \frac{2\pi}{72 f_i(\varepsilon)} < \frac{\frac{\pi}{3} - \frac{\pi}{8}}{N_{3^i-1}^{(i-1)}} < \max_{1 \leqq j \leqq g_k^{(i)}} |J_j| - 2\varkappa_i \varepsilon,$$

we get (21) for each $E_k^{(i+1)}$ $(k = 1, 2, \dots, i+1)$. Thus the statement of Lemma 3 is proved.

Lemma 4. *There exist mutually disjoint trigonometric polynomials* $S_j^{(i)}(x)$ $(j = 1, 2, \dots, 3h(i)+3;\ i = C_9, C_9+1, \dots)$[4]) *with the following properties:*

(vii) *the frequencies* ν *occurring in* $S_j^{(i)}(x)$ *satisfy* $5^{i^2} \leqq \nu \leqq 5^{i^2+1}$;

(viii) $$\int_{-\pi}^{\pi} \left(\sum_{j=1}^{3h(i)+3} S_j^{(i)}(x)\right)^2 dx \leqq \frac{C_{10}}{i+1} \qquad \left(C_{10} = C_6\left(C_8 + \frac{C_2}{8}\right)\right);$$

(ix) $$\sum_{j=\mu_1^{(i)}(x)}^{\mu_2^{(i)}(x)} S_j^{(i)}(x) \geqq \frac{1}{8} \quad \textit{for} \quad 0 \leqq x \leqq \frac{\pi}{12},$$

where $1 \leqq \mu_1^{(i)}(x) \leqq \mu_2^{(i)}(x) \leqq 3h(i)+3$.

[4]) C_9 will be defined later on, see (26).

Proof. Fix the natural number i, and apply Lemma 3 to $\varepsilon_i = 1/(i+1)$. Then we get the mutually disjoint trigonometric polynomials $U_j^{(i)}(x)$ $(j=1, 2, \ldots, h(i))$ and the simple sets $E_k^{(i)}$ $(k=1, 2, \ldots, 3^i)$. It is obvious that the frequencies occurring in $U_j^{(i)}(x)$ are smaller than $6f_i(\varepsilon_i)$, and that (17) and

$$\sum_{j=1}^{h(i)} \int_{-\pi}^{\pi} (U_j^{(i)}(x))^2\, dx \leqq C_8(i+1) \tag{23}$$

hold. In view of (iv), $E_k^{(i-1)}$ consists of $g_k^{(i-1)}$ disjoint intervals, therefore $E_k^{(i-1)(\varkappa_{i-1}\varepsilon_i)}$ consists of at most $g_k^{(i-1)}$ disjoint intervals too. Hence

$$F_i = \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - \bigcup_{k=1}^{3^i} E_k^{(i)} = \left[-\frac{\pi}{12}, \frac{\pi}{12}\right] - \bigcup_{k=1}^{3^{i-1}} E_k^{(i-1)(\varkappa_{i-1}\varepsilon_i)}$$

consists of at most $v_{i-1}(\varepsilon_i)+1$ disjoint intervals.

Let $H_i \subset [0, \pi/6]$ be the symmetric set defined by $H_i \cap [0, \pi/12] = F_i \cap [0, \pi/12]$, then H_i consists of at most $f_{i-1}(\varepsilon_i)$ disjoint intervals. Setting $H_i = \Sigma[\alpha, \beta]$, $\varepsilon_i' = \varepsilon_i/2f_{i-1}(\varepsilon_i)$ and $H_i' = [\alpha-\varepsilon_i', \beta+\varepsilon_i']$, we see that $H_i' \subset [0, \pi/6]$, $H_i' \in \mathcal{S}$ and

$$|H_i'| \leqq |H_i| + 2\varepsilon_i' f_{i-1}(\varepsilon_i) \leqq \varepsilon_i\left(1-\frac{1}{2^i}\right) + \varepsilon_i \leqq \frac{2}{i+1}.$$

Applying Lemma 1 to (H_i', ε_i', 1 and $[C_1/\varepsilon_i']$), we get the trigonometric polynomial $P^{(i)}(x)$ with frequencies 6ν $(\nu = 0, 1, \ldots, [C_1/\varepsilon_i'])$ such that

$$P^{(i)}(x) \geqq 1 \quad \text{for} \quad x \in H_i \subset H_i'^{(\varepsilon_i')} \tag{24}$$

and

$$\int_{-\pi}^{\pi} (P^{(i)}(x))^2\, dx \leqq C_2 |H_i'| \leqq \frac{2C_2}{i+1}. \tag{25}$$

Now we suppose $i \geqq C_9$ so that the inequality

$$37 f_i(\varepsilon_i) = 37 C_7^i (i+1)^i 18^{\frac{i(i-1)}{2}} = \tag{26}$$

$$= 18^{\left(\frac{1}{2}+\lambda\right)i^2 - \lambda i^2 + i\left\{\log_{18}(i+1) + \log_{18} C_7 - \frac{1}{2}\right\} + \log_{18} 37} \leqq 18^{\left(\frac{1}{2}+\lambda\right)i^2} = 5^{i^2}$$

may hold. Setting $N_1 = 5^{i^2} + 6f_i(\varepsilon_i) + (3+(-1)^i)/2$ and $N_2 = 2N_1 + 6f_i(\varepsilon_i) + 6[C_1/\varepsilon_i'] + 3$, we define

$$S_j^{(i)}(x) = (\cos N_1 x) \frac{U_j^{(i)}(x)}{i+1},$$

$$S_{h(i)+j}^{(i)}(x) = -C_3 (\cos x)(\cos N_1 x) \frac{U_j^{(i)}(x)}{i+1},$$

$$S_{2h(i)+j}^{(i)}(x) = -C_4 (\cos 2N_1 x) \frac{U_j^{(i)}(x)}{i+1}$$

$(j=1, 2, \ldots, h(i))$, and

$$S_{3h(i)+1}^{(i)}(x) = (\cos N_2 x)\frac{P^{(i)}(x)}{4},$$

$$S_{3h(i)+2}^{(i)}(x) = -C_3(\cos x)(\cos N_2 x)\frac{P^{(i)}(x)}{4},$$

$$S_{3h(i)+3}^{(i)}(x) = -C_4(\cos 2N_2 x)\frac{P^{(i)}(x)}{4}.$$

Then using (11′) we easily see that $S_j^{(i)}(x)$ $(j = 1, 2, \ldots, 3h(i)+3)$ are mutually disjoint trigonometric polynomials with frequencies ν satisfying

$$5^{i^2} \leqq N_1 - 6f_i(\varepsilon_i) - 1 \leqq \nu \leqq 2N_2 + 6\left[\frac{C_1}{\varepsilon_i'}\right].$$

And by (26),

$$2N_2 + 6\left[\frac{C_1}{\varepsilon_i'}\right] = 4\cdot 5^{i^2} + 36f_i(\varepsilon_i) + 18\left[\frac{C_1}{\varepsilon_i'}\right] + 18 \leqq 4\cdot 5^{i^2} + 37f_i(\varepsilon_i) \leqq 5^{i^2+1}.$$

By (13′), (23) and (25), we obtain

$$\int_{-\pi}^{\pi}\left(\sum_{j=1}^{3h(i)+3} S_j^{(i)}(x)\right)^2 dx =$$

$$= \sum_{j=1}^{h(i)} \int_{-\pi}^{\pi} |S_j^{(i)}(x) + S_{h(i)+j}^{(i)}(x) + S_{2h(i)+j}^{(i)}(x)|^2\, dx + \int_{-\pi}^{\pi}\left|\sum_{l=1}^{3} S_{3h(i)+l}^{(i)}(x)\right|^2 dx \leqq$$

$$\leqq \sum_{j=1}^{h(i)} C_6 \int_{-\pi}^{\pi}\left(\frac{U_j^{(i)}(x)}{i+1}\right)^2 dx + C_6\int_{-\pi}^{\pi}\left(\frac{P^{(i)}(x)}{4}\right)^2 dx \leqq$$

$$\leqq \frac{C_6}{(i+1)^2}\cdot C_8(i+1) + \frac{C_6}{16}\cdot\frac{2C_2}{i+1} = \frac{C_{10}}{i+1}.$$

To prove (ix), suppose $0 \leqq x \leqq \pi/12$. Then $x \in \bigcup_{k=1}^{3^i} E_k^{(i)}$ or $x \in H_i$. We set $\mu_1^{(i)}(x) = 1$ and $\mu_2^{(i)}(x) = m_k^{(i)}$ for

$$x \in E_k^{(i)} \cap \bigcup_{j=-\infty}^{\infty}\left[\frac{1}{N_1}\left(2j\pi - \frac{\pi}{3}\right), \frac{1}{N_1}\left(2j\pi + \frac{\pi}{3}\right)\right];$$

$\mu_1^{(i)}(x) = h(i)+1$ and $\mu_2^{(i)}(x) = h(i) + m_k^{(i)}$ for

$$x \in E_k^{(i)} \cap \bigcup_{j=-\infty}^{\infty}\left[\frac{1}{N_1}\left(2j\pi + \frac{2}{3}\right), \frac{1}{N_1}\left(2j\pi + \frac{4}{3}\pi\right)\right];$$

and $\mu_1^{(i)}(x) = 2h(i)+1$ and $\mu_2^{(i)}(x) = 2h(i)+m_k^{(i)}$ for

$$x \in E_k^{(i)} \cap \bigcup_{j=-\infty}^{\infty} \left(\frac{1}{N_1}\left(j\pi+\frac{\pi}{3}\right), \frac{1}{N_1}\left(j\pi+\frac{2}{3}\pi\right)\right).$$

Hence in the case of $x \in E_k^{(i)}$, we get

$$\sum_{j=\mu_1^{(i)}(x)}^{\mu_2^{(i)}(x)} S_j^{(i)}(x) = \begin{cases} \dfrac{\cos N_1 x}{i+1} \displaystyle\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) \\ \text{or } \dfrac{-C_3(\cos x)(\cos N_1 x)}{i+1} \displaystyle\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x) \\ \text{or } \dfrac{-C_4(\cos 2N_1 x)}{i+1} \displaystyle\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x). \end{cases}$$

Now using (14″) and (17),

$$\sum_{j=\mu_1^{(i)}(x)}^{\mu_2^{(i)}(x)} S_j^{(i)}(x) \geqq \frac{1}{i+1} \cdot \frac{\sum_{j=1}^{m_k^{(i)}} U_j^{(i)}(x)}{2} \geqq \frac{1}{8}.$$

In the case of $x \in H_i$, we set $\mu_1^{(i)}(x) = 3h(i)+1$ and $\mu_2^{(i)}(x) = 3h(i)+1$ or $3h(i)+2$ or $3h(i)+3$. Then using (14′) and (24), we get the assertion of (ix). So the proof of Lemma 4 is complete.

§ 2. Proof of the theorem

Define the sequence of natural numbers $(C_9 \leqq)m_1 < m_2 < \cdots$ such that

(27) $$\frac{\varrho(n)}{\sqrt[4]{\log_5 n}} \leqq \frac{1}{k} \quad \text{if} \quad n \geqq 5^{m_k^2}.$$

Then by (vii), setting

(28) $$T_k(x) = \sum_{j=1}^{3h(m_k)+3} S_j^{(m_k)}\left(x - \frac{(k)_{24}\pi}{12}\right) = ^{5)}$$

$$= \sum_{n=5^{m_k^2}}^{5^{m_k^2+1}} (a_n \cos nx + b_n \sin nx) = \sum_{n=5^{m_k^2}}^{5^{m_{k+1}^2}-1} (a_n \cos nx + b_n \sin nx) \qquad (k=1, 2, \dots),$$

[5]) $(k)_{24}$ denotes the remainder of k modulo 24.

we consider the series $\sum_1^\infty (a_n \cos nx + b_n \sin nx)$. And we define the rearrangement $\{n_j\}$ by

$$S_1^{(m_1)}\left(x-\frac{\pi}{12}\right)+S_2^{(m_1)}\left(x-\frac{\pi}{12}\right)+\cdots+S_{3h(m_1)+3}^{(m_1)}\left(x-\frac{\pi}{12}\right)+$$

$$+S_1^{(m_2)}\left(x-\frac{2\pi}{12}\right)+\cdots+S_{3h(m_2)+3}^{(m_2)}\left(x-\frac{2\pi}{12}\right)+\cdots+S_j^{(m_k)}\left(x-\frac{(k)_{24}\pi}{12}\right)+\cdots$$

which diverges everywhere in virtue of (ix). By (27), (28) and (viii), we get

$$\sum_{n=1}^{\infty}(a_n^2+b_n^2)\varrho^2(n) \leqq \sum_{k=1}^{\infty}\frac{\sqrt{m_k^2+1}}{k^2}\sum_{n=5^{m_k^2}}^{5^{m_k^2+1}}(a_n^2+b_n^2) =$$

$$=\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sqrt{m_k^2+1}}{k^2}\int_{-\pi}^{\pi}T_k^2(x)\,dx \leqq \frac{C_{10}}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty.$$

Thus, in accordance with the Riesz—Fischer theorem, the assertion of our theorem is proved.

Next define $(A_n \cos nx + B_n \sin nx)$ by

$$A_n=\frac{a_n\sqrt{m_k+1}}{k},\quad B_n=\frac{b_n\sqrt{m_k+1}}{k}\qquad (5^{m_k^2}\leqq n<5^{m_{k+1}^2};\ k\geqq 1),$$

and the proof of Corollary runs similarly to that of Theorem 2 in [2].

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