Hyperinvariant subspaces for spectral and n-normal operators

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If A is a bounded linear operator on a complex Banach space, then a closed linear subspace $\mathfrak{M}$ is *hyperinvariant* for A if $\mathfrak{M}$ is invariant under every operator which commutes with A. It is not known whether or not every operator other than a multiple of the identity has a non-trivial hyperinvariant subspace (i.e. other than the zero subspace and the whole space). Several sufficient conditions for the existence of non-trivial hyperinvariant subspaces are known ([3], [13], [14]).

Fuglede's theorem [6] states that every spectral subspace of a normal operator is hyperinvariant, and this was generalized to spectral operators by DUNFORD [4]. HOOVER [5] recently showed that every n-normal operator has a hyperinvariant subspace. In this note we present simple proofs of DUNFORD's and HOOVER's results, based upon Rosenblum's theorem on operator equations.

1. Rosenblum's theorem

We shall use a theorem about solutions of certain linear operator equations. The theorem was proved by ROSENBLUM [11] to the case where E and F are elements of the same Banach algebra. The result which is given below has not, to our knowledge, appeared in print before, although many people must be aware of it. Our proof is essentially the same as the proof of Rosenblum's result contained in the paper of LUMER and ROSENBLUM [9]. We denote the set of bounded inear operators from $\mathscr{X}$ to $\mathscr{Y}$ by $\mathscr{B}(\mathscr{X}, \mathscr{Y})$.

Theorem (Rosenblum). *If E and F are bounded operators on the complex Banach spaces $\mathscr{Y}$ and $\mathscr{X}$ respectively, and if the operator $\mathscr{T}$ on $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ is defined by $\mathscr{T}(X) = EX - XF$, then*

$$\sigma(\mathscr{T}) \subset \sigma(E) - \sigma(F) = \{z - w\colon z \in \sigma(E),\ w \in \sigma(F)\}.$$

Proof (similar to [9]). Define operators $\mathscr{E}$ and $\mathscr{F}$ on $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ by

$$\mathscr{E}(X) = EX \quad \text{and} \quad \mathscr{F}(X) = XF.$$

If $E-\lambda$ has in inverse, then $(E-\lambda)(E-\lambda)^{-1}X = (E-\lambda)^{-1}(E-\lambda)X = X$ for every $X\in\mathcal{B}(\mathcal{X},\mathcal{Y})$, and therefore $\sigma(\mathcal{E})\subset\sigma(E)$. Similarly $\sigma(\mathcal{F})\subset\sigma(F)$.

Since $\mathcal{E}$ and $\mathcal{F}$ are commuting operators, $\sigma(\mathcal{E}-\mathcal{F})\subset\sigma(\mathcal{E})-\sigma(\mathcal{F})$; (simply let $\mathcal{A}$ be a maximal commutative algebra containing $\mathcal{E}$ and $\mathcal{F}$ and use the fact that the spectra relative to $\mathcal{A}$, which are the ranges of the Gelfand transforms, are the same as the original spectra). Hence $\sigma(\mathcal{T})\subset\sigma(E)-\sigma(F)$.

The special case of this result that we shall need is the fact that $\sigma(E)\cap\sigma(F)=\emptyset$ and $EX_0=X_0F$ imply $X_0=0$ (since X_0 is in the nullspace of the operator $\mathcal{T}(X) = EX-XF$).

2. The Fuglede—Dunford theorem

Fuglede's theorem [6] states that whenever a bounded operator B on a Hilbert space commutes with a normal operator A, then B commutes with the spectral measure of A (or, equivalently, then B commutes with A^*). Halmos [7, 8] and Rosenblum [12] gave simplified proofs of Fuglede's theorem. Dunford [4] generalized Fuglede's theorem to the case where A is a spectral operator on a Banach space.

In this note we give another proof of Dunford's version of the theorem. We feel that this proof gives some further insight even in the Hilbert space case, although it is neither as short nor as elegant as Rosenblum's proof.

Following Dunford [4] we say that a bounded operator A on a Banach space $\mathcal{X}$ is a *spectral operator* if there exists a spectral measure $E(\cdot)$ (i.e. a countably additive mapping from the Borel sets in the complex plane into a uniformly bounded family of projections on $\mathcal{X}$ such that $E(\emptyset)=0$, $E(C)=1$, and $E(\sigma_1\cap\sigma_2)=E(\sigma_1)E(\sigma_2)$ for all Borel sets σ_1 and σ_2), which commutes with A and which has the property that $\sigma(A|E(\sigma)\mathcal{X})\subset\bar{\sigma}$ for all Borel sets σ.

Theorem (Fuglede—Dunford). *If A is a spectral operator with spectral measure $E(\cdot)$, and if $AB=BA$, then $BE(\sigma)=E(\sigma)B$ for all Borel sets σ.*

Proof. It obviously suffices to show that the range of $E(\sigma)$ is invariant under B for each Borel set σ, and this is equivalent to showing that $E(\sigma')BE(\sigma)=0$, where σ' denotes the complement of σ. By regularity it suffices to show that $E(\sigma')BE(\sigma)=0$ whenever σ is closed.

Fix a closed set σ, and let σ_0 be any closed subset of σ'. From $AB=BA$ it follows that $E(\sigma_0)AB\,E(\sigma)=E(\sigma_0)B\,AE(\sigma)$ and thus that

$$[E(\sigma_0)A\,E(\sigma_0)]\,[E(\sigma_0)B\,E(\sigma)]=[E(\sigma_0)\,B\,E(\sigma)]\,[E(\sigma)A\,E(\sigma)].$$

Hence $E(\sigma_0)B\,E(\sigma)=0$ by Rosenblum's theorem, since $E(\sigma_0)A\,E(\sigma_0)$ and

$E(\sigma)A\,E(\sigma)$ have disjoint spectra as operators on $E(\sigma_0)\mathcal{X}$ and $E(\sigma)\mathcal{X}$ respectively.

Since $E(\sigma_0)B\,E(\sigma)=0$ whenever σ_0 is a closed subset of σ', it follows that $E(\sigma')B\,E(\sigma)=0$ and the proof is complete.

3. Putnam's corollary

Soon after Fuglede's theorem was published, PUTNAM [10] observed that Fuglede's proof could be generalized to show that whenever A and C are normal operators on a Hilbert space and B is a bounded operator such that $AB=BC$ then $A^*B=BC^*$. BERBERIAN [1] found a simple trick for getting Putnam's result as a corollary of Fuglede's. To our knowledge it has not previously been observed that Berberian's trick can be applied to the case of spectral operators, yielding the following result.

Corollary. *If A and C are spectral operators with spectral measures $E(\cdot)$ and $F(\cdot)$ on the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively, and if B is a bounded operator from $\mathcal{Y}$ to $\mathcal{X}$ such that $AB=BC$, then $E(\sigma)B=BF(\sigma)$ for every Borel set σ.*

Proof. We consider $\mathcal{X}\oplus\mathcal{Y}$, with $\|(x,y)\| = \|x\|+\|y\|$, and let P and Q be the projections onto the first and second co-ordinate spaces respectively. Then the operator $T = PAP+QCQ$ is spectral, and its spectral measure is defined by $G(\sigma) = PE(\sigma)P+QF(\sigma)Q$ for each σ. A trivial computation shows that T commutes with the operator $S=PBQ$. By the Fuglede—Dunford Theorem, $G(\sigma)S=SG(\sigma)$ for each σ. Another simple computation gives $B\,E(\sigma)=F(\sigma)B$.

4. Hyperinvariant subspaces of triangular and n-normal operators

An operator is said to be *n-normal* if it is (unitarily equivalent to) an operator in the tensor product of some abelian von Neumann algebra and the algebra of $n\times n$ matrices. In other words, n-normal operators can be written in the form

$$\begin{pmatrix} A_{11} & A_{12}\dots A_{1n} \\ \vdots & \\ A_{n1} & A_{n2}\dots A_{nn} \end{pmatrix}$$

where $\{A_{ij}\}$ is a collection of commuting normal operators.

R. G. DOUGLAS and C. PEARCY showed that every 2-normal operator has a non-trivial hyperinvariant subspace, and T. B. HOOVER [5] generalized this result to n-normal operators. HOOVER shows that every n-normal operator is quasi-similar to an n-normal operator in "Jordan form", and derives the existence of hyperinvariant subspaces from this result together with Dunford's characterization of spectral operators.

We show that the existence of hyperinvariant subspaces for n-normal operators follows from the more easily proven result that every n-normal operator is unitarily equivalent to an n-normal operator in upper triangular form [2], together with the simple theorem given below.

Theorem. *If A is unitarily equivalent to an operator in the upper triangular form*

$$(*)\qquad \begin{pmatrix} A_{11} & * \dots * \\ 0 & * \dots * \\ \vdots & \\ 0 & 0 \dots A_{nn} \end{pmatrix},$$

where the spectra of A_{11} and A_{nn} are disjoint, then A has a non-trivial hyperinvariant subspace.

Proof. Let

$$B = \begin{pmatrix} * & * \dots * \\ \vdots & \vdots \\ B_{n1} & * \dots * \end{pmatrix}$$

be any operator in the commutant $\mathscr{A}$ of A. The fact that the entry in position $(n, 1)$ of AB is equal to the entry in position $(n, 1)$ of BA gives $A_{nn}B_{n1} = B_{n1}A_{11}$. Since the spectra of A_{11} and A_{nn} are disjoint, Rosenblum's theorem implies that $B_{n1} = 0$. Let x be any vector of the form $(x_1, 0, 0, 0, \dots, 0)$ with $x_1 \neq 0$ and y any vector of the form $(0, 0, \dots, 0, y_n)$ with $y_n \neq 0$. We have shown that $(Bx, y) = 0$ for all $B \in \mathscr{A}$. Thus the closure of $\{Bx : B \in \mathscr{A}\}$ is a non-trivial hyperinvariant subspace for A.

Corollary. *If A is not a multiple of the identity and is unitarily equivalent to an operator in the upper triangular form $(*)$, where A_{11} and A_{nn} are normal, then A has a non-trivial hyperinvariant subspace.*

Proof. If the spectrum of A_{11} consists of only one point, then A_{11} is a multiple of the identity. In this case A has a non-trivial eigenspace, and it is trivial to verify the fact that an eigenspace of A is hyperinvariant.

If the spectrum of A_{11} consists of more than one point, then, by the spectral theorem, we can write $A_{11} = A_{11}^0 \oplus A_{11}^1$ and $A_{nn} = A_{nn}^0 \oplus A_{nn}^1$ where the spectra of

A_{11}^0 and A_{nn}^1 are disjoint. Then A is unitarily equivalent to an operator of the form

$$\begin{pmatrix} A_{11}^0 & 0 & * & \dots & * \\ 0 & A_{11}^1 & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & & & & \\ 0 & 0 & \dots & A_{nn}^0 & * \\ 0 & & \dots & 0 & A_{nn}^1 \end{pmatrix}.$$

Thus the Theorem above gives the result.

Corollary (Hoover). *Every n-normal operator which is not a multiple of the identity has a non-trivial hyperinvariant subspace.*

Proof. A theorem of DECKARD and PEARCY [2, Theorem 2] implies that every n-normal operator is unitarily equivalent to an n-normal operator in upper triangular form. Thus the result follows from the previous corollary.

Remark. As HOOVER [5] shows, quasi-similarity preserves the existence of hyperinvariant subspaces. Thus the theorem and the first corollary above can be stated with "unitarily equivalent" replaced by "quasi-similar".

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