Operators with bounded characteristic function and their J-unitary dilation

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Introduction. Let $\mathfrak{H}$ be a (complex) Hilbert space and let T be a bounded linear operator on $\mathfrak{H}$.

Denote by Q_T the positive square root of $|I-T^*T|$ and by J_T the operator $\operatorname{sgn}(I-T^*T)$; similarly, let $Q_{T^*}=|I-TT^*|^{\frac{1}{2}}$, $J_{T^*}=\operatorname{sgn}(I-TT^*)$. Let us put

$$\Theta_T(\lambda)=[-TJ_T+\lambda Q_{T^*}(I-\lambda T^*)^{-1}Q_T]|\overline{Q_T\mathfrak{H}} \tag{0. 1}$$

whenever $(I-\lambda T^*)^{-1}$ exists. This function, whose values are operators from $\mathfrak{D}_T=\overline{Q_T\mathfrak{H}}$ to $\mathfrak{D}_{T^*}=\overline{Q_{T^*}\mathfrak{H}}$, is called the "characteristic function" of T (see [13], [10]; for the case where T is a contraction, see [15]). The main result of the present paper is the following

Theorem. *If $\Theta_T(\lambda)$ is defined for all λ with $|\lambda|<1$, and if*

$$\sup\{\|\Theta_T(\lambda)\|:|\lambda|<1\}<\infty,$$

then T is similar to a contraction.

Here similarity has the usual meaning: Two operators T, T_1 are called "similar" if there exists an affinity X (i.e. an operator mapping the space of T_1 onto the space of T in a one-to-one and continuous way) such that $T=XT_1X^{-1}$, see [15].

It is of interest to have a boundedness condition which implies similarity of T to a contraction, in view of the fact that the apparently more natural conditions

$$\sup_{n\geq 0}\|T^n\|<\infty,\quad \sup_{|\lambda|>1}(|\lambda|-1)\|(\lambda I-T)^{-1}\|<\infty,$$

formely conjectured to be sufficient for similarity to a contraction, have turned out not to be [4], [7], [8, p. 200].

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However, it is worth while to mention that the existence and boundedness of $\Theta_T(\lambda)$ on $\{\lambda: |\lambda|<1\}$ is not necessary for T being similar to a contraction. Indeed, taking $\mathfrak{H}$ the two dimensional complex Euclidean space E^2 and T the operator corresponding to the matrix $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ one obtains by simple computations that T is similar to the contraction $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, while $Q_T=Q_{T^*}=I$ and $\Theta_T(\lambda)$ is given by the matrix

$$\begin{pmatrix}(1-\lambda)^{-1} & \lambda^2(1-\lambda)^{-1}\\ 1 & \lambda\end{pmatrix}$$

which is unbounded on $\{\lambda: |\lambda|<1\}$.

The theorem is an outgrowth of two known results. The first [15, IX. 1], [6] gives the condition for a contraction to be similar to a unitary. It was generalized by L. A. SAHNOVIČ [12] to apply to general bounded T: *If $\Theta_T(\lambda)$ is defined and bounded on $\{\lambda: |\lambda|\neq 1\}$ then T is similar to a unitary operator.* Our theorem also contains the following similarity theorem of G. C. ROTA [11]: *If the spectrum $\sigma(T)$ of T is contained in $\{\lambda: |\lambda|<1\}$, then T is similar to a contraction.* Indeed $\sigma(T)\subset\{\lambda: |\lambda|<1\}$ implies that $\|(I-\lambda T^*)^{-1}\|$ is bounded on $\{\lambda: |\lambda|\leqq 1\}$ so that $\Theta_T(\lambda)$ satisfies in this case the requirements of our theorem.

Our method is the geometric interpretation of the characteristic function developed in [15, VI]. This interpretation is generalized to the case of operators which need not be contractions, by carrying forward the study of J-unitary dilation begun in [2]; but the proofs demand many considerations which did not arise for contractions. We include in § IV the proof of Sahnovič's theorem by our method.

We remark that our boundedness hypotheses are used in §§ III—IV only to ensure that we have a bounded operator on H^2, never to draw conclusions about the (operator) values which $\Theta_T(\lambda)$ assumes.

I. Preliminaries

1. As usual in this subject, it is important to note that the identity $T(I-T^*T) = (I-TT^*)T$ implies

(1. 1) $$Tf(I-T^*T) = f(I-TT^*)T$$

for any bounded complex Borel function f defined on the real line. In particular

(1. 2) $$TQ_T=Q_{T^*}T, \quad TJ_T=J_{T^*}T,$$

and taking adjoints,

(1. 2′) $$Q_TT^*=T^*Q_{T^*}, \quad J_TT^*=T^*J_{T^*}.$$

Thus $TJ_TQ_T = J_{T*}Q_{T*}T = Q_{T*}J_{T*}T$, which implies the fact already mentioned that

$$\Theta_T(\lambda)\mathfrak{D}_T \subseteq \mathfrak{D}_{T*}. \tag{1.3}$$

For later use we derive another property of the characteristic function (cf. [6, § 2]). We begin with the fact, obvious from the definition (0. 1), that

$$\Theta_T(\lambda)^* = \Theta_{T*}(\bar{\lambda}) \tag{1.4}$$

whenever either side is defined. On the other hand,

$$\Theta_T(\lambda)Q_TJ_T = Q_{T*}[-T+\lambda(I-\lambda T^*)^{-1}(I-T^*T)] = Q_{T*}(I-\lambda T^*)^{-1}(\lambda I-T).$$

Now assume that $\Theta_T(\lambda)$ and $\Theta_T(\bar{\lambda}^{-1})$ are both defined; we have

$$J_T\Theta_T(\bar{\lambda}^{-1})^*J_{T*}\Theta_T(\lambda)Q_TJ_T = J_T\Theta_{T*}(\lambda^{-1})J_{T*}Q_{T*}(I-\lambda T^*)^{-1}(\lambda I-T) =$$
$$= J_TQ_T(I-\lambda^{-1}T)^{-1}(\lambda^{-1}I-T^*)(I-\lambda T^*)^{-1}(\lambda I-T) = Q_TJ_T,$$

from which it is easy to conclude that

$$\Theta_T(\lambda)^{-1} = J_T\Theta_T(\bar{\lambda}^{-1})^*J_{T*}|\mathfrak{D}_{T*}. \tag{1.5}$$

In particular, *if $\Theta_T(\lambda)$ is defined and bounded on $\{\lambda\colon |\lambda|\neq 1\}$ then $\Theta_T(\lambda)^{-1}$ exists and is bounded on* $D=\{\lambda\colon |\lambda|<1\}$; thus in this case

$$\sup_D \|\Theta_T(\lambda)\| < \infty, \quad \sup_D \|\Theta_T(\lambda)^{-1}\| < \infty. \tag{1.6}$$

2. We now recall the construction of the J-unitary dilation [2]. The present discussion differs somewhat in notation, and deals only with bounded T.

The dilation will be an operator on a direct sum space

$$\mathfrak{K} = \cdots \oplus \overset{(-1)}{\mathfrak{D}_{T*}} \oplus \overset{(0)}{\mathfrak{H}} \oplus \overset{(1)}{\mathfrak{D}_T} \oplus \overset{(2)}{\mathfrak{D}_T} \oplus \cdots. \tag{1.7}$$

This means that there are canonical injections of $\mathfrak{H}$, $\mathfrak{D}_{T*}$, and $\mathfrak{D}_T$ onto orthogonal subspaces of $\mathfrak{K}$. We indicate these injections by supercript indices. Thus for any $h\in\mathfrak{H}$, $\overset{(0)}{h}$ denotes the corresponding element of the 0-th co-ordinate subspace $\overset{(0)}{\mathfrak{H}}$ of $\mathfrak{K}$; for any $h\in\mathfrak{D}_{T*}$, $\overset{(i)}{h}$ denotes the corresponding element of the i-th co-ordinate subspace $\overset{(i)}{\mathfrak{D}_{T*}}$ of $\mathfrak{K}$ ($i=-1, -2, \ldots$); and for any $h\in\mathfrak{D}_T$, $\overset{(i)}{h}$ denotes the corresponding element of the i-th co-ordinate subspace $\overset{(i)}{\mathfrak{D}_T}$ of $\mathfrak{K}$ ($i=1,2,\ldots$). The general element of $\mathfrak{K}$ is a sequence $\sigma=(h_i)_{i=-\infty}^{\infty}$ with $h_0\in\mathfrak{H}$, $h_i\in\mathfrak{D}_{T*}$ $(i<0)$, $h_i\in\mathfrak{D}_T$ $(i>0)$, and $\|\sigma\|^2 = \sum_{i=-\infty}^{\infty}\|h_i\|^2 < \infty$; we can equally well write σ as a sum

$$\sigma = \sum_{i=-\infty}^{\infty} \overset{(i)}{h_i} \tag{1.8}$$

of elements in co-ordinate subspaces of $\mathfrak{K}$.

We define operators U and J on $\mathfrak{K}$ by specifying how they act on the above general element σ:

$$U\sigma = \sum_{i\neq 0,1} \overset{(i)}{h_{i-1}} + \overset{(0)}{h'} + \overset{(1)}{h''}, \tag{1.9}$$

$$h' = Q_{T^*}h_{-1} + Th_0, \quad h'' = -T^*J_{T^*}h_{-1} + Q_T h_0; \tag{1.9'}$$

$$J\sigma = \sum_{i<0} \overset{(i)}{(J_{T^*}h_i)} + \overset{(0)}{h_0} + \sum_{i>0} \overset{(i)}{(J_T h_i)}. \tag{1.10}$$

Then $J^* = J = J^{-1}$, and U is J-unitary, i.e.,

$$(JU\sigma, U\sigma') = (J\sigma, \sigma') \qquad (\sigma, \sigma' \in \mathfrak{K}) \tag{1.11}$$

and U is invertible; we shall have need for the explicit expression for its inverse, acting upon the general σ of (1.8):

$$U^{-1}\sigma = \sum_{i\neq -1,0} \overset{(i)}{h_{i+1}} + \overset{(-1)}{k'} + \overset{(0)}{k''}, \tag{1.12}$$

$$k' = J_{T^*}Q_{T^*}h_0 - J_{T^*}Th_1, \quad k'' = T^*h_0 + J_T Q_T h_1. \tag{1.12'}$$

U is a dilation of T, that is, for all $h \in \mathfrak{H}$,

$$\overset{(0)}{(T^n h)} = PU^n \overset{(0)}{h} \qquad (n = 0, 1, 2, \ldots), \tag{1.13}$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\overset{(0)}{\mathfrak{H}}$. We obtain a J-isometric dilation of T (i.e., an operator satisfying the analogues of (1.11) and (1.13), but not necessarily invertible) if we consider the restriction U_+ of U to a certain invariant subspace $\mathfrak{K}_+$. Namely, $\mathfrak{K}_+ = \bigvee_{n\geq 0} U^n \overset{(0)}{\mathfrak{H}}$, or, perhaps more simply, $\mathfrak{K}_+$ is the set of all vectors $\sum_{i=0}^{\infty} \overset{(i)}{h_i}$ in $\mathfrak{K}$. Evidently $\mathfrak{K}_+$ reduces J.

3. We conclude the preliminaries by recalling some well-known simple notions about geometry of subspaces of Hilbert spaces and J-spaces, which are central to our main arguments below. These will be stated in a general context: Let $\mathfrak{M}$ and $\mathfrak{N}$ be any subspaces of any Hilbert space $\mathfrak{H}$, and let P and Q respectively be the orthoprojectors onto $\mathfrak{M}$ and $\mathfrak{N}$. Then we have (see e.g. [1])

Lemma 1.1. *The operators $PQ|\mathfrak{M}$ and $QP|\mathfrak{N}$ have the same spectrum, except perhaps for* 0.

Let us say that $\mathfrak{M}$ is "not far from" $\mathfrak{N}$ in case $0 \notin \sigma(PQ|\mathfrak{M})$. (In more conventional terminology [3, 1], $\mathfrak{M}$ neither intersects nor is asymptotic to $\mathfrak{H} \ominus \mathfrak{N}$.) If A denotes $Q|\mathfrak{M}$ as an operator from $\mathfrak{M}$ to $\mathfrak{N}$, then $A^*A = PQ|\mathfrak{M}$; thus $\mathfrak{M}$ is not far from $\mathfrak{N}$ if and only if there exists $c > 0$ such that, for all $m \in \mathfrak{M}$, $\|Qm\| \geq c\|m\|$. A necessary and sufficient condition that $\mathfrak{M}$ be not far from $\mathfrak{N}$ and $\mathfrak{N}$ not far from $\mathfrak{M}$ is that $Q|\mathfrak{M}$ be an invertible map of $\mathfrak{M}$ onto $\mathfrak{N}$.

Lemma 1.2. *If $\mathfrak{M}$ is not far from $\mathfrak{N}$ then $\mathfrak{H}\ominus\mathfrak{N}$ is not far from $\mathfrak{H}\ominus\mathfrak{M}$.*

This follows immediately from the previous Lemma: $0\notin\sigma(PQ|\mathfrak{M})$ implies $1\notin\sigma(P(1-Q)|\mathfrak{M})$, which implies $1\notin\sigma((1-Q)P|\mathfrak{H}\ominus\mathfrak{N})$, which implies

$$0\notin\sigma((1-Q)(1-P)|\mathfrak{H}\ominus\mathfrak{N}),\text{ q.e.d.}$$

See also [14, Lemma 9. 1. 1].

Lemma 1.3. *If $\mathfrak{M}$ is not far from $\mathfrak{N}$ then $\mathfrak{M}+(\mathfrak{H}\ominus\mathfrak{N})$ is closed (and is the direct sum of $\mathfrak{M}$ and $\mathfrak{H}\ominus\mathfrak{N}$).*

This is well known, e.g. [9, § 3], [3, I].

Now let there also be defined on $\mathfrak{H}$ a symmetry J, i.e. $J^{-1}=J=J^*$, making it a J-space. We will use the notion of a regular subspace (pravil'noe podprostranstvo) of $\mathfrak{H}$ [5]. Let $\mathfrak{M}$ and P be as above; let P_+ denote $\frac{1}{2}(I+J)$, the orthoprojector onto the canonical positive subspace of $\mathfrak{H}$. $\mathfrak{M}$ is called "regular" in case it is not far from $J\mathfrak{M}$, in the sense defined above.

Using the fact that the orthoprojector onto $J\mathfrak{M}$ is JPJ, and that $PJPJ|\mathfrak{M}$ is the square of the hermitian operator $PJ|\mathfrak{M}$, it is not hard to see that each of the following conditions is equivalent to $\mathfrak{M}$ being regular:

(i) $\|PJx\|$ defines on $\mathfrak{M}$ a norm equivalent to the given norm;
(ii) $PJ|\mathfrak{M}$ has a bounded inverse on $\mathfrak{M}$;
(iii) $\frac{1}{2}\notin\sigma(PP_+|\mathfrak{M})$;
(iv) $\frac{1}{2}\notin\sigma(P_+P|P_+\mathfrak{H})$.

The equivalence of (iii) with (iv) here is a case of Lemma 1. 1.

Lemma 1.4. *If $\mathfrak{M}$ is regular, then the following are also regular: $J\mathfrak{M}$; the orthogonal complement $\mathfrak{H}\ominus\mathfrak{M}$ of $\mathfrak{M}$; and the J-orthogonal complement $\mathfrak{H}\ominus J\mathfrak{M}$ of M.*

As to $J\mathfrak{M}$, this follows from (i) and the fact that J is unitary; as to $\mathfrak{H}\ominus\mathfrak{M}$, it follows from (iv); the rest is obvious.

It is only for regular subspaces that the J-orthogonal complement deserves its name:

Lemma 1.5. *If $\mathfrak{M}$ is regular, then $\mathfrak{H}$ is the direct sum of $\mathfrak{M}$ and $\mathfrak{H}\ominus J\mathfrak{M}$.*

This is a corollary of Lemma 1. 3. (The converse is known too, but we will not need it.)

We now return to the special context of the Introduction, so the symbols $\mathfrak{H}$, J, etc. will have the special meanings which were attached to them.

II. The characteristic function and the J-unitary dilation

1. We will now show that the dilation contruction gives rise to the characteristic function here in almost as natural a way as in the case of contractions.

For this purpose we consider two subspaces on which U_+ acts as a unilateral shift (of some multiplicity $\geqq 0$). First,

$$\mathfrak{K}_+ = \overset{(0)}{\mathfrak{H}} \oplus \mathfrak{M}, \quad \mathfrak{M} = \bigoplus_{i=1}^{\infty} \overset{(i)}{\mathfrak{D}_T} = \bigvee_{n \geqq 0} U^n \overset{(1)}{\mathfrak{D}_T}, \tag{2.1}$$

and $U_+|\mathfrak{M}$ is, by definition, an isometric mapping of each co-ordinate subspace onto the next.

Second, we consider

$$\mathfrak{M}_* = \bigvee_{n \geqq 0} U^{n+1} \overset{(-1)}{\mathfrak{D}_{T^*}}. \tag{2.2}$$

It is plain from (1.9), (1.9′) that $\mathfrak{M}_* \subseteqq \mathfrak{K}_+$. In the contractive case, it was shown [15] that in (2.2) as well, U_+ maps each of the sequence of subspaces isometrically onto the next. In the general case, it need not be isometric, but it is expansive: for all

$$\sigma = \sum_{i=0}^{\infty} \overset{(i)}{h_i} \in \mathfrak{K}_+$$

we have

$$\|U_+\sigma\|^2 = \|Th_0\|^2 + \|Q_T h_0\|^2 + \|h_1\|^2 + \cdots \geqq$$

$$\geqq \|Th_0\|^2 + (J_T Q_T h_0, Q_T h_0) + \|h_1\|^2 + \cdots = \|h_0\|^2 + \|h_1\|^2 + \cdots = \|\sigma\|^2.$$

2. Let us now introduce the Fourier representations of $\mathfrak{M}$ and $\mathfrak{M}_*$. For finite sums

$$\sigma = \sum_{n=0}^{N} \overset{(n+1)}{h_n} = \sum_{n=0}^{N} U^n \overset{(1)}{h_n} \in \mathfrak{M}, \qquad \sigma_* = \sum_{n=0}^{N_*} U^{n+1} \overset{(-1)}{h_{*n}} \in \mathfrak{M}_* \tag{2.3}$$

(where $h_n \in \mathfrak{D}_T$, $h_{*n} \in \mathfrak{D}_{T^*}$), we put

$$\Phi\sigma(\lambda) = \sum_{n=0}^{N} \lambda^n h_n, \quad F\sigma_*(\lambda) = \sum_{n=0}^{N_*} \lambda^n h_{*n} \quad (|\lambda| < 1). \tag{2.4}$$

Linear applications are thereby defined from dense subsets of $\mathfrak{M}$, resp. $\mathfrak{M}_*$, into the space $H^2(\mathfrak{D}_T)$, resp. $H^2(\mathfrak{D}_{T^*})$. These are Hardy H^2 spaces of vector-valued functions, see [15, V]. The mapping Φ is obviously isometric and can be extended to a unitary mapping of $\mathfrak{M}$ onto $H^2(\mathfrak{D}_T)$, which will still be denoted by Φ. Under this isomorphism, the isometric unilateral shift $U_+|\mathfrak{M}$ corresponds to Λ: $\Phi U_+|\mathfrak{M} = \Lambda\Phi$. Here Λ is the multiplication by the independent variable, that is, for $u \in H^2(\mathfrak{D}_T)$ we have $\Lambda u(\lambda) = \lambda u(\lambda)$ $(|\lambda| < 1)$. This correspondence of unilateral shift to multiplication is the essential feature of the Fourier representation. It carries over to the non-isometric Fourier representation F: if Λ_* denotes the multiplication by λ in $H^2(\mathfrak{D}_{T^*})$ then obviously $FU\sigma_* = \Lambda_* F\sigma_*$ for the above finite sums σ_*.

We introduce J-space structure in the H^2 spaces in the natural way. Denote by $\mathbf{J}$ the operator defined on $H^2(\mathfrak{D}_T)$ by $(\mathbf{J}u)(\lambda) = J_T(u(\lambda))$ $(|\lambda| < 1)$. It is immediate that $\Phi J|\mathfrak{M} = \mathbf{J}\Phi$ and hence identically $(\mathbf{J}\Phi\sigma, \Phi\sigma) = (J\sigma, \sigma)$ $(\sigma \in \mathfrak{M})$, showing how to regard Φ as preserving also the J-space structure. Similarly, define $\mathbf{J}_*$ on $H^2(\mathfrak{D}_{T^*})$ by $(\mathbf{J}_* u_*)(\lambda) = J_{T^*}(u_*(\lambda))$. We will verify the relation

$$(2.5) \qquad (\mathbf{J}_* F\sigma_*, F\sigma_*) = (J\sigma_*, \sigma_*)$$

for finite sums in $\mathfrak{M}_*$, but it is less immediate because the terms in the definition (2. 3) of σ_* do not belong to subspaces which are clearly invariant under J. However, the J-unitary property (1. 11) of U allows us to write

$$(J\sigma_*, \sigma_*) = \sum_{n=0}^{N_*} \sum_{m=0}^{N_*} (JU^{n+1} \overset{(-1)}{h_{*n}}, U^{m+1} \overset{(-1)}{h_{*m}}) = \sum_{n=0}^{N_*} (J \overset{(-1)}{h_{*n}}, \overset{(-1)}{h_{*n}}) = \sum_{n=0}^{N_*} (J_{T^*} h_{*n}, h_{*n})$$

(the terms for $m \neq n$ vanish because $U^{m-n} \overset{(-1)}{\mathfrak{D}_{T^*}} \perp \overset{(-1)}{\mathfrak{D}_{T^*}}$). But the right-hand member, by the definition of J_* and the definition of the inner product in H^2, is equal to $(\mathbf{J}_* F\sigma_*, F\sigma_*)$, with $F\sigma_*$ as in (2. 4). Thus (2. 5) is proved.

3. We thus have two naturally defined subspaces $\mathfrak{M}$ and $\mathfrak{M}_*$, and the projectors $P_{\mathfrak{M}}$, $P_{\mathfrak{M}^*}$ onto them do not commute. It is not surprising that fairly complete information about T is contained in an invariant description of the contraction $P_{\mathfrak{M}^*}|\mathfrak{M}$. If one tries to make this description giving $\mathfrak{M}$ and $\mathfrak{M}_*$ their Fourier representations, one finds the contraction from $\mathfrak{M}$ to $\mathfrak{M}_*$ is replaced by a mapping from $H^2(\mathfrak{D}_T)$ to $H^2(\mathfrak{D}_{T^*})$, given exactly by the characteristic function.

We now exhibit this relationship formally, for arbitrary T. In the following section we will give it a geometric sense, by using the hypothesis of boundedness of the characteristic function.

For any $u \in H^2(\mathfrak{D}_T)$, with power-series expansion $u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$, let $\Theta_T u$ denote the function whose values are defined by $(\Theta_T u)(\lambda) = \Theta_T(\lambda) u(\lambda)$. This function is defined and analytic, with values in $\mathfrak{D}_{T^*}$, at least for $|\lambda| < \min(\|T\|^{-1}, 1)$, and this is all we need for the moment (indeed it would be possible to proceed using only formal power series). We can write in the neighborhood of $\lambda = 0$

$$(2.6) \qquad (\Theta_T u)(\lambda) = \sum_{n=0}^{\infty} \lambda^n \left(\sum_{m=0}^{n} \theta_{n-m} u_m \right);$$

here the θ_n are the Taylor coefficients of $\Theta_T : \Theta_T(\lambda) = \sum_{n=0}^{\infty} \lambda^n \theta_n$. From the definition

(0. 1) we can derive this explicit expression:

$$(2.7)\qquad (JU^m \overset{(1)}{h}, U^{n+1} \overset{(-1)}{h_*}) = \begin{cases} 0 & \text{if } n<m, \\ (\theta_{n-m} J_T h, h_*) & \text{if } n \geqq m \end{cases}$$

for all $h \in \mathfrak{D}_T$, $h_* \in \mathfrak{D}_{T^*}$. To prove this, use the J-unitary property in the same way as above:

$$(JU^m \overset{(1)}{h}, U^{n+1} \overset{(-1)}{h_*}) = (JU^{m-n-1} \overset{(1)}{h}, \overset{(-1)}{h_*})$$

which obviously is 0 for $m \geqq n+1$. If $m-n-1 = -k$, $k>0$, then we need to find the component of $U^{-k} \overset{(1)}{h}$ in $\overset{(-1)}{\mathfrak{D}_{T^*}}$; this we can do by iterating (1. 12), (1. 12′), and the result is

$$-J_{T^*} Th \quad \text{if} \quad k=1, \qquad J_{T^*} Q_{T^*} T^{*k-2} J_T Q_T h \quad \text{if} \quad k>1.$$

Therefore

$$(JU^{-k} \overset{(1)}{h}, \overset{(-1)}{h_*}) = \begin{cases} (-Th, h_*) = (\theta_0 J_T h, h_*) & \text{if} \quad k=1, \\ (Q_{T^*} T^{*k-2} Q_T J_T h, h_*) = (\theta_{k-1} J_T h, h_*) & \text{if} \quad k>1 \end{cases}$$

using (0. 1). This establishes (2. 7).

We are now in a position to discuss inner products of elements of $\mathfrak{M}$ with elements of $\mathfrak{M}_*$. Let σ, σ_* be as in (2. 3). Then, by (2. 7),

$$(2.8)\qquad (J\sigma, \sigma_*) = \sum_{m=0}^{N} \sum_{n=0}^{N_*} (JU^m \overset{(1)}{h_m}, U^{n+1} \overset{(-1)}{h_{*n}}) = \sum_{n \geqq m \geqq 0} (\theta_{n-m} J_T h_m, h_{*n}).$$

Also by (2. 6) and (2. 4)

$$(\Theta_T \Phi J\sigma)(\lambda) = (\Theta_T J_T \Phi\sigma)(\lambda) = \sum_{n=0}^{\infty} \lambda^n \left(\sum_{m=0}^{\min(n,N)} \theta_{n-m} J_T h_m \right).$$

This is analitic in λ with values in $\mathfrak{D}_{T^*}$, but need not lie in $H^2(\mathfrak{D}_{T^*})$; if it does, its inner product with $F\sigma_*$ from (2. 4) is, by (2. 8),

$$\sum_{n=0}^{N_*} \left(\sum_{m=0}^{\min(n,N)} \theta_{n-m} J_T h_m, h_{*n} \right) = (J\sigma, \sigma_*).$$

III. Geometric properties of the J-unitary dilation in case the characteristic function is bounded

1. Assume now that Θ_T is defined on the open unit disk D and that

$$\sup_D \|\Theta_T(\lambda)\| = C < \infty.$$

Then for any $u \in H^2(\mathfrak{D}_T)$, $\Theta_T u$ belongs to $H^2(\mathfrak{D}_{T^*})$ and its norm in that space is $\leqq C\|u\|$. Let $\Theta: H^2(\mathfrak{D}_T) \to H^2(\mathfrak{D}_{T^*})$ be defined by $\Theta u = \Theta_T u$; it is an operator of norm C.

The conclusion of the last section can now be rewritten as

$$(J\sigma, \sigma_*) = (\Theta\Phi J\sigma, F\sigma_*).$$

Because elements $J\sigma$ (with σ a finite sum (2.3)) are dense in $\mathfrak{M}$, and Θ and Φ are continuous, we deduce that

$$(3.1) \qquad (\mu, \sigma_*) = (\Theta\Phi\mu, F\sigma_*) \qquad (\mu \in \mathfrak{M}).$$

This is not quite the promised interpretation of Θ_T in terms of $P_{\mathfrak{M}^*}$ because the second factor in the inner product is still restricted to be a finite sum.

We will remedy this by proving *that F has a unique extension to an affinity of $\mathfrak{M}_*$ onto $H^2(\mathfrak{D}_{T^*})$.*

To this end taking $\mu = P_{\mathfrak{M}}\sigma_*$ in (3.1), we obtain

$$\|P_{\mathfrak{M}}\sigma_*\|^2 = (P_{\mathfrak{M}}\sigma_*, \sigma_*) = (\Theta\Phi P_{\mathfrak{M}}\sigma_*, F\sigma_*) \leqq C\|P_{\mathfrak{M}}\sigma_*\| \, \|F\sigma_*\|$$

whence

$$(3.2) \qquad \|P_{\mathfrak{M}}\sigma_*\|^2 \leqq C^2\|F\sigma_*\|^2.$$

Let P denote the projection onto the complement of $\mathfrak{M}$ in $\mathfrak{K}_+$, which by (2.1) is $\overset{(0)}{\mathfrak{H}}$. By (2.5) and the definition (1.10) of J,

$$(\mathbf{J}_* F\sigma_*, F\sigma_*) = (J\sigma_*, \sigma_*) = \|P\sigma_*\|^2 + (JP_{\mathfrak{M}}\sigma_*, P_{\mathfrak{M}}\sigma_*),$$

which yields, because $\mathbf{J}_*$ and J are contractions,

$$\|P\sigma_*\|^2 \leqq \|P_{\mathfrak{M}}\sigma_*\|^2 + \|F\sigma_*\|^2 \leqq (C^2+1)\|F\sigma_*\|^2$$

(using (3.2)). Add this to (3.2) to obtain

$$\|\sigma_*\|^2 \leqq (1+2C^2)\|F\sigma_*\|^2.$$

This proves that F has a bounded inverse G. The domain of G is dense, so G has a unique bounded extension to the whole of $H^2(\mathfrak{D}_{T^*})$; denote this also by G. By continuity, we deduce from (2.5) that

$$(JGu, Gu) = (\mathbf{J}_* u, u) \qquad (u \in H^2(\mathfrak{D}_{T^*})).$$

This is the same as saying $G^* P_{\mathfrak{M}_*} JG = \mathbf{J}_*$ $(= \mathbf{J}_*^{-1})$. Thus G has the left-inverse $\mathbf{J}_* G^* P_{\mathfrak{M}^*} J$, which as a product of bounded operators is bounded. It is an extension of F because F is inverse to G on a dense set. This completes the proof that F has a unique extension to an affinity; the extension will still be denoted by F.

Then we know also that σ_* in (3.1) can be replaced by an arbitrary element μ_* of $\mathfrak{M}_*$.

2. We now introduce the residual part of U, in imitation of the contraction case. The images of $\overset{(-1)}{\mathfrak{D}}_{T^*}$ under non-negative powers of U^{-1} do span all of $\mathfrak{K} \ominus \mathfrak{K}_+$.

Its images under positive powers of U, on the other hand, span the subspace $\mathfrak{M}_*$ which need not be all of $\mathfrak{K}_+$. Consider the J-orthogonal complement of $\mathfrak{M}_*$:

$$\mathfrak{R} = \mathfrak{K}_+ \ominus J\mathfrak{M}_* = \mathfrak{K} \ominus J \bigvee_{-\infty}^{\infty} U^n \overset{(-1)}{\mathfrak{D}}_{T^*}; \tag{3. 3}$$

it is clear that the two definitions are equivalent. The latter expression (3. 3), together with the J-unitary property of U (1. 11), make it clear that $\mathfrak{R}$ is invariant under both U and U^{-1}. Thus we may define the "residual part" $R = U|\mathfrak{R}$, an invertible operator.

Being a restriction of U_+ (not just of U), R is expansive (see § II.1 above). Hence

$$\|R^{-1}\| \leqq 1. \tag{3. 4}$$

Our next aim is to prove that

$$\sup_{-\infty<n<\infty} \|R^n\| < \infty, \tag{3. 5}$$

and (3. 4) takes care of this for all $n \leqq 0$.

Return to (2. 5), which implies at once

$$F^* \mathbf{J}_* F = P_{\mathfrak{M}^*} J | \mathfrak{M}_*.$$

Now that we are able to assert that F (and therefore also F^*) is an affinity, we can deduce that the equation represents an invertible operator on $\mathfrak{M}_*$. That is, $\mathfrak{M}_*$ is a regular subspace of the J-space $\mathfrak{K}_+$. (See § I. 3.) By Lemma 1. 4, we deduce now from (3. 3) that $\mathfrak{R}$ is also regular, that is, that $P_{\mathfrak{R}} J|\mathfrak{R}$ is invertible.

Set $J_{\mathfrak{R}} = P_{\mathfrak{R}} J|\mathfrak{R}$. We now know that for some $c > 0$

$$c\|\varrho\| \leqq \|J_{\mathfrak{R}}\varrho\| \leqq \|\varrho\| \qquad (\varrho \in \mathfrak{R}). \tag{3. 6}$$

But we also know from the remarks following (3. 3) that

$$(J_{\mathfrak{R}} R^{-1}\varrho, R^{-1}\varrho') = (JU^{-1}\varrho, U^{-1}\varrho') = (J\varrho, \varrho') = (J_{\mathfrak{R}}\varrho, \varrho')$$

for $\varrho, \varrho' \in \mathfrak{R}$, so that (iterating) $J_{\mathfrak{R}} = (R^{-n})^* J_{\mathfrak{R}} R^{-n}$ $(n > 0)$. With (3. 4) and (3. 6), this gives

$$c\|\varrho\| \leqq \|J_{\mathfrak{R}}\varrho\| = \|(R^{-n})^* J_{\mathfrak{R}} R^{-n}\varrho\| \leqq \|J_{\mathfrak{R}} R^{-n}\varrho\| \leqq \|R^{-n}\varrho\|,$$

whence $\|R^n\varrho\| \leqq \dfrac{1}{c}\|\varrho\|$ $(\varrho \in \mathfrak{R};\ n = 1, 2, \ldots)$.

To sum up, (3. 5) has been established, with

$$\sup_{n>0} \|R^n\| \leqq \frac{1}{c}; \quad \sup_{n<0} \|R^n\| \leqq 1.$$

Now we appeal to the theorem of B. Sz.-Nagy that any operator R with

$\sup_{-\infty<n<\infty} \|R^n\| \leqq \frac{1}{c} < \infty$ is similar to a unitary [14]. More precisely, it tells us that *there exists a self-adjoint invertible operator A on $\mathfrak{R}$ such that*

$$\|A\| \cdot \|A^{-1}\| \leqq \frac{1}{c}$$

and such that $V = A^{-1}RA$ is unitary.

3. We are ready to prove the theorem stated in the introduction. We begin by defining a new Hilbert space

$$\mathbf{H} = H^2(\mathfrak{D}_{T^*}) \oplus \mathfrak{R}$$

with a canonical mapping into $\mathfrak{K}_+$:

$$X(u \oplus \varrho) = F^{-1}u + A\varrho \qquad (u \in H^2(\mathfrak{D}_{T^*}),\ \varrho \in \mathfrak{R}). \tag{3.7}$$

As u and ϱ vary, the term $F^{-1}u$ here ranges over all of $\mathfrak{M}_*$ and the term $A\varrho$ over all of $\mathfrak{R}$, because F^{-1} and A are affinities. But $\mathfrak{R}$ is the J-orthogonal complement of $\mathfrak{M}_*$ by definition (3. 3), and $\mathfrak{M}_*$ was just proved to be regular, so by Lemma 1. 5, X maps $\mathbf{H}$ onto $\mathfrak{K}_+$.

Let P again denote the orthoprojector on $\mathfrak{K}_+$ onto $\overset{(0)}{\mathfrak{H}}$; we now see that PX maps $\mathbf{H}$ onto $\overset{(0)}{\mathfrak{H}}$. Let Q denote the orthoprojector on $\mathbf{H}$ onto the orthogonal complement of the null-space of PX. We define Y: $Q\mathbf{H} \to \mathfrak{H}$ by

$$Y(u \oplus \varrho) = h \text{ if and only if } PX(u \oplus \varrho) = \overset{(0)}{h}. \tag{3.8}$$

Being continuous, 1—1, and onto, Y must be an affinity of $Q\mathbf{H}$ onto $\mathfrak{H}$.

Now the operator $\mathbf{U}$ defined by

$$\mathbf{U}(u \oplus \varrho) = \Lambda_* u \oplus V\varrho,$$

where V is the unitary found in § III. 2, is an isometry on $\mathbf{H}$; and it is related to U_+ by the application (3. 7):

$$X\mathbf{U} = F^{-1}\Lambda_* + AV = U_+ F^{-1} + RA = U_+ X. \tag{3.9}$$

We project down onto $\overset{(0)}{\mathfrak{H}}$. That is, we operate on (3. 9) on the left by P; using the definition (3. 8) and the dilation property (1. 13), we obtain

$$YQ\mathbf{U} = TY.$$

But Y is an affinity and $Q\mathbf{U}$ is certainly a contraction (on $Q\mathbf{H}$ to $Q\mathbf{H}$). This completes the proof of the theorem.

IV. Similarity to a unitary operator

This section will be devoted to the proof of the result of SAHNOVIČ stated in the introduction. Accordingly we now strengthen the hypotheses used in § III, and assume that $\Theta_T(\lambda)$ is defined for $|\lambda| \neq 1$ and

$$\sup_{|\lambda| \neq 1} \|\Theta_T(\lambda)\| = C < \infty.$$

We saw in § I. 1 that this makes $\Theta_T(\lambda)$ and $\Theta_T(\lambda)^{-1}$ both uniformly bounded analytic operator-functions on D, see (1. 5) and (1. 6). Therefore Θ is an affinity of $H^2(\mathfrak{D}_T)$ onto $H^2(\mathfrak{D}_{T^*})$; indeed its inverse is given by

$$(\Theta^{-1}u_*)(\lambda) = \Theta_T(\lambda)^{-1}u_*(\lambda) \qquad (|\lambda| < 1)$$

for $u_* \in H^2(\mathfrak{D}_{T^*})$.

We begin, as before, with (3. 1), extended to

$$(\mu, \mu_*) = (\Theta\Phi\mu, F\mu_*) \qquad (\mu \in \mathfrak{M},\ \mu_* \in \mathfrak{M}_*),$$

$$P_{\mathfrak{M}^*}|\mathfrak{M} = F^*\Theta\Phi. \tag{4. 1}$$

Now, however, since all three operators on the right are affinities, we are able to short-cut the considerations of § III. 3. Indeed, (4. 1) says directly that $P_{\mathfrak{M}_*}|\mathfrak{M}$ is an affinity of $\mathfrak{M}$ onto $\mathfrak{M}_*$. This implies that $\mathfrak{M}$ is not far from $\mathfrak{M}_*$ and $\mathfrak{M}_*$ is not far from $\mathfrak{M}$, in the sense of § I. 3. By Lemma 1. 2, $\overset{(0)}{\mathfrak{H}}$ is not far from $J\mathfrak{R}$ and vice versa. Applying the unitary J, we see that $J\overset{(0)}{\mathfrak{H}}$ $(=\overset{(0)}{\mathfrak{H}})$ is in the same relationship to $\mathfrak{R}$. Hence $P|\mathfrak{R}$ is an affinity of $\mathfrak{R}$ onto $\overset{(0)}{\mathfrak{H}}$ just as in the contraction case.

Let A, V be the operators found in § III. 2. Define Y: $\mathfrak{R} \to \mathfrak{H}$ by

$$Y\varrho = h \quad \text{if and only if} \quad PA\varrho = \overset{(0)}{h}.$$

Then Y is an affinity from $\mathfrak{R}$ onto $\mathfrak{H}$; and the equation

$$PAV = PRA = PUA,$$

together with the dilation relation (1. 13), gives $YV = TY$. This is a similarity of T to a unitary operator, as was required.

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