Some absolute topological properties under monotone unions

By TINUOYE M. ADENIRAN in Zaria (Nigeria)*)

1. Definition. A property P is said to be *absolute under monotone unions* (aumu) *in a class* $\mathscr{C}$ of topological spaces if, for any given Y and X_i $(i=1, 2, \ldots)$ in $\mathscr{C}$, with $X_i \subset Y$, $X_i \subset X_{i+1}$, the fact that each X_i has property P implies that $\bigcup_{i=1}^{\infty} X_i$ also has property P.

Connectedness and arcwise (path) connectedness are absolute under monotone unions in the class $\mathscr{C}_a$ of all topological spaces. But local connectedness and disconnectedness are not so; as an example illustrating the former, consider the Warsaw circle W, consisting of the curve $\sin \frac{\pi}{x}$ $(0<x\leqq 1)$, the interval $(-1, +1)$ of the y axis and a simple curve joining the points $(0, -1)$ and $(1, 0)$. Take as X_n the set

$$W - \left\{(x, y): y = \sin\frac{\pi}{x}, \quad 0 < x \leqq \frac{\pi}{n}\right\};$$

then each X_i is locally connected but $\bigcup_{i=1}^{\infty} X_i = W$ is not so. For an example illustrating the latter, let P be the set of irrationals in E^1 and let $Q=\{r_1, r_2, \ldots\}$ be an enumeration of $E^1 - P$. Let $P_j = P \cup \{r_1, r_2, \ldots, r_j\}$. Each P_i is disconnected, but $\bigcup_{i=1}^{\infty} P_i = E^1$ is not. This last example also shows that the property of being 0-dimensional is not aumu in the class of all topological spaces.

By restricting $\mathscr{C}_a$ to the class $\mathscr{C}_0$ of countable metric spaces, disconnectedness is aumu in $\mathscr{C}_0$. This is a simple consequence of the well-known fact, that any non-void connected metric space has at least a continuum number of points.

A further example for a property which is aumu is the property of being F_σ in the class $\mathscr{C}_a$. But the property of being G_δ is not aumu in $\mathscr{C}_a$. This is well known, nevertheless we shall give a simple counter-example:

Consider the real line E^1. A finite set of rationals is trivially G_δ, but *the set Q of rationals is not G_δ in E^1*. This follows easily from the Baire category theorem and from the fact that Q is a set of first category in itself.

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2. The reader can easily see that a monotone union of T_0-spaces is a T_0-space. In this section we shall show that the property of being T_1 is also aumu in any class $\mathscr{C}$ while any separation axiom beyond this is not. We state the first assertion as

Theorem 1. *The property of being a T_1-space is aumu in any class $\mathscr{C}$.*

Proof. Let Y be a topological space with the sequence $\{X_i: X_i \subset X_{i+1}\}$ of subsets of Y such that each X_i is T_1. Let $X = \bigcup_{i=1}^{\infty} X_i$ and let x, y be two distinct points of X. Then there exists, for some $j \in Z^+$, $X_j \subset X$ such that $x, y \in X_j$. Since X_j is T_1 there exist open sets U', V' in X_j such that $x \in U'$, $x \notin V'$, $y \in V'$, $y \notin U'$. Furthermore there exist open sets U, V in X such that $U \cap X_j = U'$ and $V \cap X_j = V'$. Since $x \in U'$, $x \in U$, similarly $y \in V$. $x \in X_j$ and $x \notin V'$ imply that $x \notin V$, similarly $y \notin U$. We have thus found sets U, V open in X with $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$. By definition, this entails that X is T_1 and the theorem is proved.

Corollary. *Let $\{X_i: X_i \subset X_{i+1}\}$ be a sequence of spaces such that each X_i is T_2 (regular, Tychonoff, normal). Then $\bigcup_{i=1}^{\infty} X_i$ is at least T_1.*

Theorem 2. *The property of being T_2 is not aumu in $\mathscr{C}_a$.*

Proof. Let I be the open unit interval (0, 1), and let

$$(1) \qquad X_k = (0 \times I) \cup (1 \times I) \cup \left(\frac{1}{2} \times I\right) \cup \cdots \cup \left(\frac{1}{k} \times I\right) \quad \text{for} \quad k = 1, 2, \ldots .$$

Each X_k is a finite union of open intervals in E^2 and since each I is T_2, each X_k is also T_2. So let $X = \bigcup_{i=1}^{\infty} X_i$. Topologize X as follows: On $X - (0 \times I)$ use the usual topology on E^1. For a neighbourhood of a point x in $0 \times I$, take an open interval in I of $0 \times I$ containing x and all the $\left(\frac{1}{j} \times I\right)$ with $\frac{1}{j} < \varepsilon$, where ε is an arbitrary positive real. Now let P_1 and P_2 be two distinct points of $0 \times I$. It is easy to see that any open subset of X containing P_1 meets any other containing P_2; we cannot therefore have two disjoint open sets containing P_1 and P_2 respectively; hence $X = \bigcup_{i=1}^{\infty} X_i$ is not T_2. — The same example shows:

Theorem 3. *Any separation axiom implying T_2 is not aumu in $\mathscr{C}_a$.*

Each X_k in (1) is metrizable, but X is not normal, and therefore not metrizable. So we have:

Theorem 4. *Metrizability is not aumu in $\mathscr{C}_a$.*

References

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