D'Alembert's functional equation in Banach algebras

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1. Suppose B is a Banach algebra and $f\colon R\to B$ (R denotes the field of real numbers) such that

(1) $$f(s+t)+f(s-t) = 2f(s)f(t)$$

for all $s, t\in R$. S. KUREPA [6] has shown that if B has identity e, $f(0)=e$, and f is measurable then there exists a unique $b\in B$ such that

$$f(s) = e+\frac{s^2b}{2!}+\frac{s^4b^2}{4!}+\cdots$$

for all $s\in R$. Note that if $b=a^2$ for some $a\in B$ then $f(s) = \frac{1}{2}\{\exp(sa)+\exp(-sa)\} = \cosh(sa)$ for all $s\in R$. In this paper we consider the problem of finding the solutions of (1) on $(0, \infty)$ and without the assumption that B has an identity. The main result is that if $f\colon (0,\infty)\to B$ satisfies (1) for $s>t>0$ and if $\lim_{t\to 0+} f(t)$ exists then there exists $j, b, c\in B$ such that $j^2=j$, $jb=bj=b$, $cj=c$, $jc=0$ and $f(s) = \left(j+\frac{s^2b}{2!}+\frac{s^4b^2}{4!}+\cdots\right)+c\left(sj+\frac{s^3b}{3!}+\frac{s^5b^2}{5!}+\cdots\right)$ for all $s>0$. This result is analogous to a result concerning the functional equation $f(s+t) = f(s)f(t)$ which can be found on page 283 of the book of HILLE and PHILLIPS [4]. Also included in the present paper are certain general results concerning (1) when the domain is an Abelian group and the range is an associative algebra over the rationals. Some regularity properties are also included in cases when topologies are present.

2. We begin by deriving some general properties of solutions of (1). Let G be an additive Abelian group, let B be an associative algebra over the field of rational numbers and suppose $f\colon G\to B$ satisfies (1) for all $s, t\in G$.

Let $j=f(0)$. Then, putting $s=t=0$ in (1) we find

(2) $$j^2=j.$$

With $t=0$ in (1) we have

(3) $$f(s)=f(s)j$$

for all $s\in G$.

Now let g and h be the even and odd parts of f respectively; that is, $2g(s) = f(s)+f(-s)$, $2h(s) = f(s)-f(-s)$ for all $s\in G$. Letting $s=0$ in (1) we find

$$g=jf. \tag{4}$$

Thus $g = jg+jh$ and so, since g and jg are even,

$$jh=0. \tag{5}$$

From (4) and (2) it follows that

$$jg=j^2f=jf=g. \tag{6}$$

Now (3) implies

$$gj=g \tag{7}$$

and

$$hj=h. \tag{8}$$

Thus, by (7) and (5),

$$g(s)h(t)=(g(s)j)h(t)=g(s)(jh(t))=0 \tag{9}$$

and similarly, by (5) and (8),

$$h(s)h(t)=0 \tag{10}$$

for all $s, t\in G$. Using (4), (1) and (9) we conclude that

$$\begin{aligned} g(s+t)+g(s-t) &= j(f(s+t)+f(s-t)) = 2jf(s)f(t) = 2g(s)f(t) = \\ &= 2g(s)g(t)+2g(s)h(t) = 2g(s)g(t) \quad \text{for all} \quad s, t\in G. \end{aligned} \tag{11}$$

If $f(0)=0$ then, by (3), $f\equiv 0$. If $j\neq 0$ then j is an identity for the subalgebra $B'=\{x\in B\colon jx=xj=x\}$ and, from (6) and (7), $g(s)\in B'$ for all $s\in G$. Thus g can be considered as a mapping of G into B' which is a solution of (11), or (1) and $g(0)=j$, the identity of B'.

From (9) and (10) we find

$$\begin{aligned} h(s+t)+h(s-t) &= f(s+t)+f(s-t)-g(s+t)-g(s-t) = \\ &= 2f(s)f(t)-2g(s)g(t) = 2h(s)g(t) \quad \text{for all} \quad s, t\in G. \end{aligned} \tag{12}$$

3. In this section we impose topologies on G and B and consider some regularity properties of solutions of (1).

Proposition 1. *Let G be a locally compact Abelian group, let B be a Banach algebra and suppose $f\colon G\to B$ satisfies* (1) *for all $s, t\in G$. If f is strongly measurable on a set of positive, finite Haar measure, then the mapping $t\to f(2t)$ is continuous at* 0.

Proof. Suppose f is strongly measurable on a measurable set A of positive finite Haar measure. Then f is the pointwise limit almost everywhere on A of a sequence of countably valued measurable functions (see [4] page 72). As in the complex valued case, the theorems of Egorov and Lusin can be proved (see [3] pages 158—160) and we conclude that there exists a compact subset K of A of positive Haar measure such that the restriction of f to K is continuous. It follows that f is uniformly continuous on K. (See [7] page 256.)

Since K has positive finite Haar measure there exists a neighborhood V of $0 \in G$ such that

$$K \cap (K+v) \cap (K-v) \neq \emptyset$$

whenever $v \in V$. (See [2] page 296.)

Let $\varepsilon > 0$ and $M = \max \{\|f(t)\| : t \in K\}$. Since f is uniformly continuous on K there exists a symmetric neighborhood U of $0 \in G$ such that $\|f(s)-f(t)\| < \varepsilon/4M$ provided $s, t \in K$ and $s-t \in U$. Now

$$f(2v)+f(2u) = 2f(u+v)f(u-v)$$

and so

$$\|f(2v)-f(0)\| = 2\|f(u+v)f(u+v)-f(u)f(u)\| \leqq$$

$$\leqq 2\|f(u+v)\| \, \|f(u-v)-f(u)\| + 2\|f(u)\| \, \|f(u+v)-f(u)\|.$$

If $v \in V \cap U$ then there exists $u \in K$ such that $u+v \in K$ and $u-v \in K$ so that $v \in V \cap U$ implies $\|f(2v)-f(0)\| < \varepsilon$.

Corollary. *If in addition to the hypotheses of Proposition 1 it is assumed that the mapping $t \to 2t$ is a bicontinuous automorphism of G, then f is continuous at* 0.

Proposition 2. *Let X be a Hausdorff linear topological space, B a Banach algebra and suppose $f: X \to B$ satisfies* (1) *for all $s, t \in X$. If f is continuous at* 0, *then f is continuous everywhere.*

Proof. Replace s by nt in (1) where n is a positive integer to find that

$$f((n+1)t) = 2f(nt)f(t)-f((n-1)t)$$

for all $t \in X$ and $n = 1, 2, \dots$. Since f is continuous at 0, f is bounded on an open neighborhood U of $0 \in X$. Hence, by induction, f is bounded on nU for $n = 1, 2, 3, \dots$. But $X = \bigcup_{n=1}^{\infty} nU$ and thus f is bounded in a neighborhood of each point of X since each nU is open. We know that

$$\lim_{t \to 0} \frac{f(s+t)+f(s-t)}{2} = \lim_{t \to 0} f(s)f(t) = f(s)f(0) = f(s)$$

for all $s \in X$ by (1) and (3). Suppose f is not continuous at some fixed $s \in X$. Then there exists $d>0$ and a net $\{t_\alpha\}\subset X$ such that $t_\alpha\to 0$ and

$$\|f(s+t_\alpha)-f(s)\| \geqq d \quad \text{for all} \quad \alpha.$$

But then, by (1) and (3),

$$\|f(s+2t_\alpha)-f(s)\| = \|f(s+2t_\alpha)+f(s)-2f(s+t_\alpha)-2f(s)+2f(s+t_\alpha)\| =$$
$$= \|\{2f(s+t_\alpha)f(t_\alpha)-2f(s+t_\alpha)f(0)\}-2\{f(s)-f(s+t_\alpha)\}\| \geqq$$
$$\geqq 2\|f(s)-f(s+t_\alpha)\|-2\|f(s+t_\alpha)\{f(t_\alpha)-f(0)\}\|$$

for all α. Since f is bounded in a neighborhood of s and f is continuous at 0, $\lim_\alpha \|f(s+t_\alpha)\{f(t_\alpha)-f(0)\}\| = 0$. Hence

$$\limsup_\alpha \|f(s+2t_\alpha)-f(s)\| \geqq 2d.$$

It follows by induction that

$$\limsup_\alpha \|f(s+2^k t_\alpha)-f(s)\| \geqq 2^k d$$

for each $k=1, 2, \ldots$ which contradicts the fact that f is bounded in a neighborhood of s. Thus, by contradiction, f is continuous at every $s\in X$.

Corollary. *If B is a Banach algebra, $f\colon R^n\to B$ satisfies* (1) *for all $s, t\in R^n$ and if f is measurable on a set of positive, finite, n-dimensional Lebesgue measure, then f is continuous.*

Proof. This follows from the corollary to Proposition 1 and Proposition 2.

4. The theorem of this section, which generalizes a theorem of S. Kurepa [6], is the main result of this paper. In its proof we use several properties of a Riemann-type integral for vector valued functions for which we omit the elementary proofs. If $[a, b]$ is a compact interval, if X is a Banach space and $f\colon [a, b]\to X$ is continuous, then f is uniformly continuous on $[a, b]$. As in the real valued case one can prove the existence of a unique $x\in X$ which has the following property: To each $\varepsilon>0$ there corresponds $\delta>0$ such that $\|x-\sum_{k=1}^{n}(t_k-t_{k-1})f(s_k)\| < \varepsilon$ provided $a=t\leqq \leqq s_1\leqq t_1\leqq s_2\leqq t_2\leqq\cdots\leqq t_{n-1}\leqq s_n\leqq t_n=b$ and $|t_k-t_{k-1}| < \delta$ for $k=1, 2, \ldots, n$. We write $x = \int_a^b f(t)dt$ and call this vector the integral of f over $[a, b]$.

Lemma. *Let X be a Banach space and let $0<a<\infty$. Suppose that $\varphi\colon (0,a)\to X$ is continuous, $\varphi'(t)$ exists, and $\|\varphi'(t)\|\leqq M<\infty$ for $0<t<a$. Then*

(i) $\lim_{t\to 0+}\varphi(t)=\alpha$ *exists;*

(ii) *if* $\lim_{t\to 0+}\varphi'(t)=\beta$ *exists, we have* $\beta = \lim_{t\to 0+}\frac{1}{t}(\varphi(t)-\alpha)$.

Proof. (i) Suppose $\{t_n\}_{n=1}^{\infty} \subseteqq (0, a)$ and $t_n \to 0$ as $n \to \infty$. Then

$$\|\varphi(t_n) - \varphi(t_m)\| = \left\| \int_{t_n}^{t_m} \varphi'(t)\, dt \right\| \leqq M|t_n - t_m| \to 0 \quad \text{as} \quad n, m \to \infty.$$

Thus $\alpha = \lim_{n\to\infty} \varphi(t_n)$ exists since X is complete. If $\{s_n\}_{n=1}^{\infty} \subseteqq (0, a)$ and $s_n \to 0$ as $n \to \infty$ then $\alpha' = \lim_{n\to\infty} \varphi(s_n)$ exists. Letting $u_n = t_n$ for n even and $u_n = s_n$ for n odd we find

$$\alpha = \lim_n \varphi(t_n) = \lim_n \varphi(u_n) = \lim_n \varphi(s_n) = \alpha'.$$

Hence $\lim_{t\to 0+} \varphi(t)$ exists and is equal to α.

(ii) Let $\Phi(t) = \begin{cases} \varphi'(t) & \text{if } 0 < t < a, \\ \beta & \text{if } t = 0. \end{cases}$ Then Φ: $[0, a) \to X$ is continuous and

$$\int_0^s \Phi(t)\, dt = \int_0^\varepsilon \Phi(t)\, dt + \int_\varepsilon^s \varphi'(t)\, dt = \int_0^\varepsilon \Phi(t)\, dt + \varphi(s) - \varphi(\varepsilon)$$

whenever $0 < \varepsilon < s < a$. Letting $\varepsilon \to 0+$ we conclude $\varphi(s) - \alpha = \int_0^s \Phi(t)dt$ for $0 < s < a$ and so

$$\frac{1}{s}(\varphi(s) - \alpha) = \frac{1}{s} \int_0^s \Phi(t)\, dt \to \Phi(0) = \beta \text{ as } s \to 0+.$$

Theorem. *Let B be a Banach algebra and let f: $(0, \infty) \to B$ be such that*

$$f(s+t) + f(s-t) = 2f(s)f(t)$$

whenever $s > t > 0$. If $\lim_{t\to 0+} f(t) = j$ exists then $j^2 = j$ and there exist elements $b, c \in B$ such that $jb = bj = b$, $cj = c$, $jc = 0$ and

$$f(s) = \left(j + \frac{s^2 b}{2!} + \frac{s^4 b^2}{4!} + \cdots +\right) + c\left(sj + \frac{s^3 b}{3!} + \frac{s^5 b^2}{4!} + \cdots\right) \tag{13}$$

for all $s > 0$. Conversely, with such j, b, and c, if f is defined by (13) *for all $s \in R$ then f satisfies* (1) *for all $s, t \in R$.*

Proof. We begin by proving the first assertion. Putting $s = 2t$ in (1) we find

$$f(3t) + f(t) = 2f(2t)f(t) \tag{14}$$

for all $t > 0$. If we let $t \to 0+$ in (14) we conclude that $j^2 = j$.

Since $\lim_{t\to 0+} f(t)$ exists, f is bounded on an interval of the form $(0, a)$ for some $a > 0$. But then (14) implies f is bounded on $(0, (3/2)a)$. By induction one can prove that f is bounded on any finite subinterval of $(0, \infty)$.

We now aim to show that

$$f(t)j=f(t) \tag{15}$$

for all $t>0$. To this end let $\varphi(t)=f(t)-f(t)j$ for $t>0$. Since $j^2=j=\lim_{t\to 0+} f(t)$ we have $\lim_{t\to 0+}\varphi(t)=0$. Also, whenever $s>t>0$, $\varphi(s+t)+\varphi(s-t)=2f(s)f(t)-2f(s)f(t)j=2f(s)\varphi(t)$. If $u>v>0$,

$$\varphi(u)+\varphi(v)=2f\left(\frac{u+v}{2}\right)\varphi\left(\frac{u-v}{2}\right).$$

Fix $a>0$ and let $M=\sup\{\|f(t)\|:0<t<a\}$. Let $\varepsilon>0$ and choose $\delta>0$ such that $0<t<\delta$ implies $\|\varphi(t)\|<\varepsilon/4M$. Then if $0<v<u<a$ and $u-v<2\delta$,

$$\|\varphi(u)+\varphi(v)\| \leqq 2M(\varepsilon/4M)=\varepsilon/2$$

so that

$$\|\varphi(u)-\varphi(v)\|=\left\|\varphi(u)+\varphi\left(\frac{u+v}{2}\right)-\varphi(v)-\varphi\left(\frac{u+v}{2}\right)\right\| \leqq$$

$$\leqq \left\|\varphi(u)+\varphi\left(\frac{u+v}{2}\right)\right\|+\left\|\varphi(v)+\varphi\left(\frac{u+v}{2}\right)\right\|<\varepsilon.$$

We have shown that φ is uniformly continuous on $(0,a)$ for any $a>0$ and hence φ is continuous. Thus, for any $s>0$,

$$2\varphi(s)=\lim_{t\to 0+}\varphi(s+t)+\varphi(s-t)=\lim_{t\to 0+}2f(s)\varphi(t)=0,$$

which proves (15).

The next step in the proof consists of showing that f is continuous. Let $a>0$ and $M=\{\|f(t)\|: 0<t<a\}$. If $0<v<u<a$ then by (1) and (15)

$$\left\|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right\|=\left\|2f\left(\frac{u+v}{2}\right)f\left(\frac{u-v}{2}\right)-2f\left(\frac{u+v}{2}\right)j\right\| \leqq$$

$$\leqq 2M\left\|f\left(\frac{u-v}{2}\right)-j\right\|.$$

Thus for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\left\|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right\|<\varepsilon \tag{16}$$

whenever $0<u$, $v<a$ and $0<|u-v|<\delta$.

Now suppose f is not continuous at s where $0<s<a$. Then there exist $d>0$ and a sequence $\{t_n\}$ converging to 0 such that $\|f(s+t_n)-f(s)\| \geqq d$ for each $n=1,2,\dots$.

Hence

$$\| f(s+2t_n)-f(s)\| = \| f(s+2t_n)+f(s)-2f(s+t_n)+2f(s+t_n)-2f(s)\| \geqq$$
$$\geqq 2\| f(s+t_n)-f(s)\| - \| f(s+2t_n)+f(s)-2f(s+t_n)\|$$

for each $n=1, 2, \dots$. But, by (16),

$$\lim_{n\to\infty} \|f(s+2t_n)+f(s)-2f(s+t_n)\| = 0$$

so that

$$\limsup_{n\to\infty} \|f(s+2t_n)-f(s)\| \geqq 2d.$$

As in the proof of Proposition 2, this contradicts the boundedness of f in a neighborhood of s. Thus f is continuous at s. Since a was arbitrary, f is continuous on $(0, \infty)$.

Now define $F(s)=\begin{cases} f(s) & \text{for } s>0 \\ j & \text{for } s=0 \end{cases}$. Then F is continuous on $[0, \infty)$,

(17) $$F(s+t)+F(s-t) = 2F(s)F(t)$$
whenever $s\geqq t\geqq 0$ and
(18) $$Fj=F.$$

Motivated by the consideration in section 2 we let $G=jF$ and $H = F-G$. Then
(19) $$G(0)=jF(0)=j^2=j \quad \text{and} \quad H(0) = F(0)-G(0) = 0,$$

G and H are continuous on $[0, \infty)$ and, by (18),

(20) $$jG=G=Gj, \quad jH=0 \quad \text{and} \quad Hj=H.$$
Therefore, by (20),
(21) $$G(s)H(t)=(G(s)j)H(t)=G(s)(jH(t))=0$$
and
(22) $$H(s)H(t)=(H(s)j)H(t)=H(s)(jH(t))=0 \quad \text{for all} \quad s, t\geqq 0.$$

Let $B'=\{x\in B\colon xj=jx=x\}$. Then B' is a closed subalgebra of B and is thus a Banach algebra. Furthermore, j is the identity of B'. Also note that, by (20), $G\colon [0, \infty)\to B'$ and, from (21),

(23) $$G(s+t)+G(s-t) = 2jF(s)F(t) = 2G(s)G(t)$$
provided $s\geqq t\geqq 0$.

Let $a>0$. If $0<\varepsilon<a<s$ then, by (23),

$$\int_0^\varepsilon G(s+t)+G(s-t)\,dt = 2G(s)\int_0^\varepsilon G(t)\,dt.$$

But $\lim_{\varepsilon\to 0+}(1/\varepsilon)\int_0^\varepsilon G(t)\,dt = G(0) = j$ so for sufficiently small $\varepsilon > 0$, $\int_0^\varepsilon G(t)\,dt$ has an inverse in B'. We fix $\varepsilon > 0$ and let $\gamma^{-1} = \int_0^\varepsilon G(t)\,dt$ to deduce that $G(s) = \frac{1}{2}\left\{\int_s^{s+\varepsilon} G(t)\,dt - \int_{s-\varepsilon}^{s} G(t)\,dt\right\}\gamma$ for all $s > a$. It follows that G has continuous derivatives of every order on (a, ∞) and, since a was arbitrary, G has continuous derivatives of every order on $(0, \infty)$.

Differentiating (23) with respect to t we find

$$G'(s+t) - G'(s-t) = 2G(s)G'(t)$$

whenever $s > t > 0$. With sufficiently small $s > 0$,

$$\lim_{t\to 0+} G'(t) = \lim_{t\to 0+}\frac{1}{2}G(s)^{-1}[G'(s+t) - G'(s-t)] = 0.$$

By the lemma,

$$G'(0) = \lim_{t\to 0+}\frac{G(t) - G(0)}{t} = 0. \tag{24}$$

From (23) it follows that

$$G''(s+t) + G''(s-t) = 2G(s)G''(t) \tag{25}$$

for $s > t > 0$. Thus for sufficiently small $s > 0$

$$\lim_{t\to 0+} G''(t) = \lim_{t\to 0+}\frac{1}{2}[G(s)^{-1}][G''(s+t) + G''(s-t)] = G(s)^{-1}G''(s).$$

It follows from the lemma that G' is continuously differentiable on $[0, \infty)$. If we let $b = G''(0) \in B'$ and let $t \to 0+$ in (25) we find that

$$G''(s) = G(s)b \tag{26}$$

for all $s > 0$. Since $b \in B'$, (26) also holds if $s = 0$.

From (26), (24) and (19) it follows that

$$G(t) = j + \int_0^t \int_0^u G(s)b\,ds\,du = j + \int_0^t (t-s)G(s)b\,ds$$

for all $t \geqq 0$. By iteration one finds

$$G(t) = j + \frac{t^2 b}{2!} + \cdots + \frac{t^{2n} b^n}{(2n)!} + \frac{1}{(2n+1)!}\int_0^t (t-s)^{2n+1} G(s) b^{n+1}\,ds$$

for all $t \geqq 0$. The last term on the right tends to 0 as $n \to \infty$ for any fixed $t > 0$, so

$$(27) \qquad G(t) = j + \frac{t^2 b}{2!} + \frac{t^4 b^2}{4!} + \cdots$$

for all $t \geqq 0$ since this series converges absolutely. Also note that $bj = jb = b$ since $b \in B'$.

We now solve for H. From (17) and (23),

$$H(s+t) + H(s-t) = 2F(s)F(t) - 2G(s)G(t)$$

and then, in view of (21) and (22), we find

$$(28) \qquad H(s+t) + H(s-t) = 2H(s)G(t) \quad \text{for} \quad s \geqq t \geqq 0.$$

As with G, we deduce from (28) that H has continuous derivatives of every order on $(0, \infty)$. Differentiating (27) twice with respect to t and letting $t \to 0+$ we find

$$(29) \qquad H''(s) = H(s)b \quad \text{for all} \quad s > 0.$$

Now since $\lim_{s\to 0+} H''(s) = \lim_{s\to 0+} H(s)b = 0$ it follows from the lemma that $\lim_{s\to 0+} H'(s) = c$ exists. Another application of the lemma proves that $H'(0) = c$ exists and $c = \lim_{s\to 0+} H'(s)$.

As with G, we deduce from (28), (19), and the fact that $H'(0) = c$ that for all $s > 0$

$$(30) \qquad H(s) = c\left(sj + \frac{s^3 b}{3!} + \frac{s^5 b^2}{5!} + \cdots\right).$$

From (20) we find that $jc = 0$ and $cj = c$.

We have thus shown that f satisfies (13) for all $s > 0$.

To prove the converse let $j, b, c \in B$ such that $jb = bj = b$, $cj = c$ and $jc = 0$. Define $G: R \to B$ by (27) and $H: R \to B$ by (30) and let $f(s) = G(s) + H(s)$ for all $s \in R$. Note that $bc = (bj)c = b(jc) = 0$ and thus

$$(31) \qquad G(s)H(t) = H(s)H(t) = 0$$

for all $s, t \in R$. It is not difficult to verify directly that G satisfies (23) for all $s, t \in R$. Note that $H = cG'$ so that

$$(32) \quad H(s+t) + H(s-t) = cG'(s+t) + cG'(s-t) = 2cG'(s)G(t) = 2H(s)G(t)$$

for all $s, t \in R$. Thus by (23), (31) and (32)

$$f(s+t) + f(s-t) = 2G(s)G(t) + 2H(s)G(t) =$$
$$= 2[G(s) + H(s)][G(t) + H(t)] = 2f(s)f(t) \quad \text{for all } s, t \in R.$$

This completes the proof of the theorem.

The following corollary follows directly from the corollary to Proposition 2 and the above theorem.

Corollary. *Let B be a Banach algebra and suppose f: $R \to B$ is such that* (1) *is true for all $s, t \in R$. Then f is measurable on a set of positive Lebesgue measure if and only if f has the form* (13) *for constants $j, b, c \in B$ satisfying $j^2 = j$, $jb = bj = b$, $cj = c$ and $cj = 0$.*

Remarks. Many authors have considered equation (1), often called D'Alembert's equation (see [1]). KANNAPPAN [5] has shown that the general solution of (1) among complex valued functions defined on an Abelian group G is of the form $f(s) = \frac{1}{2} \{m(s) + m(-s)\}$ where m is a complex valued function defined on G and satisfying $m(s+t) = m(s)m(t)$ for all $s, t \in G$. SOVA [8] has considered the strongly continuous solutions of (1) where f is defined on $(0, \infty)$ and has values in the Banach algebra of bounded operators on a Banach space and succeeded in proving an analogue of the Hille—Yosida theorem in the theory of semi-groups of operators.

References

[1] J. ACZÉL, *Lectures on Functional Equations and Their Applications* (New York, 1966).
[2] E. HEWITT and K. A. ROSS, *Abstract Harmonic Analysis* (New York, 1963).
[3] E. HEWITT and K. STROMBERG, *Real and Abstract Analysis* (New York, 1965).
[4] E. HILLE and R. S. PHILLIPS, *Functional Analysis and Semi-groups* (Providence, 1957).
[5] PL. KANNAPPAN, The functional equation $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc.*, **19** (1968), 69—74.
[6] S. KUREPA, A cosine functional equation in Banach algebras, *Acta Sci. Math.*, **23** (1962), 255—267.
[7] W. RUDIN, *Fourier Analysis on Groups* (New York, 1962).
[8] M. SOVA, Cosine operator functions, *Rosprawy Mat.*, **49** (1966), 3—46.

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