# A connection between commutativity and separation of spectra of operators

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**1. Introduction.** Recent results indicate that there is a basic connection between the commutativity of certain operators on a Banach space and the spectra of those operators. In [2] it was shown that if $A$ is an operator on a complex Banach space and $\sigma(A)\cap\sigma(e^{2\pi ik/n}A)=\emptyset$ for $k=1,\dots,n-l$, then $A$ and $A^n$ commute with the same operators. This result was strongly generalized in [3] as follows: if $f$ is holomorphic on a neighborhood of $\sigma(A)$, $f$ is $1-1$ on $\sigma(A)$ and $f'(z)\neq 0$ on $\sigma(A)$, then $A$ and $f(A)$ commute with the same operators. In this paper we generalize the results of [2] for the case $n=2$ by considering two operators $A$ and $B$ such that $\sigma(A)\cap\sigma(B)=\emptyset$.

**2. Notation and terminology.** We shall consider a Banach algebra $\mathscr{B}$ with an identity element $I$ and elements $A, B, X, \dots$; $\sigma(A)$ is the *spectrum* of $A$. In case $\mathscr{B}$ is the algebra of continuous linear operators on a Hilbert space we use the standard notation: if $A\in\mathscr{B}$, then $A^*$ is the (Hilbert space) adjoint of $A$, $\operatorname{Re} A=(A+A^*)/2$, and $\operatorname{Im} A=(A-A^*)/2i$. In this case we say that $A$ is *normal* if $AA^*=A^*A$ and $A$ is *unitary* if $AA^*=A^*A=I$.

**3. The theorem.** In [4, Theorem 3.1] it was proved that if $\sigma(A)\cap\sigma(B)=\emptyset$, then for each $Y$ in $\mathscr{B}$ there exists a unique solution to the equation $BX-XA=Y$. In particular, $BX-XA=0$ only in case $X=0$. We use this result to prove:

Theorem. *If $\sigma(A)\cap\sigma(B)=\emptyset$, then $X$ commutes with each of $A$ and $B$ if and only if $X$ commutes with each of $A+B$ and $AB$.*

Proof. One of the implications is obvious. Assume that $X$ commutes with $A+B$ and $AB$. Then

$$A(AX-XA)-(AX-XA)B = A^2X-AX(A+B)+X(AB) =$$

$$= A^2X-A(A+B)X+(AB)X = 0.$$

Thus by [4, Theorem 3. 1], we have $AX-XA=0$. It is now obvious that $BX-XB=0$ also.

The hypothesis of the theorem, calling for a separation of the spectrum of $A$ and the spectrum of $B$, is dictated by the example of the operators $A=-B=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ on two-dimensional complex Banach space. In this case $A+B=AB=0$.

**4. Applications.** We list below a few of the general applications of our theorem and then concentrate on the applications to operators on Hilbert space.

Corollary 1. *If $\sigma(A)\cap\sigma(B)=\emptyset$, then $A$ and $B$ commute if and only if $A+B$ and $AB$ commute.*

Proof. $A$ and $B$ commute if and only if $A+B$ commutes with each of $A$ and $B$. Apply the theorem with $X=A+B$.

Corollary 2. ([2] and [3]) *If $\sigma(A)\cap\sigma(-A)=\emptyset$, then $X$ commutes with $A$ if and only if $X$ commutes with $A^2$.*

Proof. Apply the theorem with $B=-A$.

The next result is applicable to any invertible element of $\mathscr{B}$ of norm less than 1.

Corollary 3. *If $A$ is invertible and $\sigma(A)\cap\sigma(A^{-1})=\emptyset$, then $X$ commutes with $A$ if and only if $X$ commutes with $A+A^{-1}$.*

Proof. Apply the theorem with $B=A^{-1}$.

Other general algebraic applications are obvious.

In Corollaries 4—8 we assume that $\mathscr{B}$ is the Banach algebra of continuous linear operators on a Hilbert space.

The first application to operators on Hilbert space is obtained by choosing $B=A^*$.

Corollary 4. *If $\sigma(A)\cap\sigma(A^*)=\emptyset$, then $X$ commutes with each of $A$ and $A^*$ if and only if $X$ commutes with each of* $\operatorname{Re} A$ *and* $AA^*$.

A special result of Corollary 4 is obtained by choosing $X=\operatorname{Re} A$.

Corollary 5. *If $\sigma(A)\cap\sigma(A^*)=\emptyset$, then $A$ is normal if and only if* $\operatorname{Re} A$ *commutes with $AA^*$.*

This last corollary is reminiscent of the result in [1, Theorem 1]: $A$ is normal if and only if each of $AA^*$ and $A^*A$ commutes with $\operatorname{Re} A$. The restriction on the spectrum of $A$ in Corollary 5 thus reduces the number of commutativity relations required to force $A$ to be normal.

Another consequence of Corollary 4 is obtained by assuming that $A$ is unitary.

Corollary 6. *If $A$ is unitary and $\sigma(A) \cap \sigma(A^*) = \emptyset$, then $X$ commutes with $A$ if and only if $X$ commutes with* $\operatorname{Re} A$.

If the Hilbert space under consideration is finite dimensional and $A$ is any unitary operator, then it follows from Corollary 6 that some unit multiple of $A$, say $e^{i\theta}A$, is such that an operator $X$ commutes with $\operatorname{Re}(e^{i\theta}A)$ if and only if $X$ commutes with $\operatorname{Im}(e^{i\theta}A)$.

Corollary 7. *If either* $\operatorname{Re} A$ *or* $\operatorname{Im} A$ *is invertible, then $X$ commutes with each of $A$ and $A^*$ if and only if $X$ commutes with each of $A$ and* $(\operatorname{Re} A)\cdot(\operatorname{Im} A)$.

Proof. Under the hypothesis, we have $\sigma(\operatorname{Re} A) \cap \sigma(i \operatorname{Im} A) = \emptyset$. Apply the theorem with $A_1 = \operatorname{Re} A$ and $B_1 = i \operatorname{Im} A$.

As a final application consider Corollary 7 with $X = A$ to give an equivalent condition for the normality of an operator $A$.

Corollary 8. *If either* $\operatorname{Re} A$ *or* $\operatorname{Im} A$ *is invertible, then $A$ is normal if and only if $A$ commutes with* $(\operatorname{Re} A)\cdot(\operatorname{Im} A)$.

## References

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