On the Suzuki structure theory for non self-adjoint operators on Hilbert space

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Throughout this paper all Hilbert spaces will be complex and all operators considered on them will be linear and bounded. Let A be an operator and $p(z, \bar{z})$ a complex non-commutative polynomial in z and $\bar{z}$. In Section 1 we shall give a complete structure theorem for the operator A whenever $p(A, A^*)$ is compact. The theorems in Section 1 are based on the structure of the W^*-algebra generated by A and they will include the results of N. SUZUKI [14], who developed this theory for an operator A with $\operatorname{Im} A$ compact, and also the generalizations of Suzuki's work by H. BEHNCKE [1] and [2] and the author [8]. In Section 2 we shall give an application of this theory to the study of non self-adjoint spectral operators on Hilbert space. By using C^*-algebra techniques, one can also obtain many of the results in this paper. In particular, Lemma 4 in [1] and its generalization to non-separable spaces play a role in the C^*-algebra development analogous to the role of Proposition 1 in the W^*-algebra approach presented here.

If A is an operator on a Hilbert space, we shall denote by $R(A)$ the W^*-or von Neumann algebra generated by A, that is, the smallest weakly closed algebra containing A and I and closed under the operation of taking adjoints. The set of all operators which commute with every operator in $R(A)$ is called the commutant of $R(A)$ and is denoted by $R(A)'$. N. SUZUKI [14] called an operator *primary* if $R(A)$ is a factor, that is, if its center $Z(A) \equiv R(A) \cap R(A)'$ consists of the scalar multiples of the identity. For the terminology, notation and basic results on von Neumann algebras we refer to J. DIXMIER [6].

[1]) This paper was prepared while the author was an Office of Naval Research Postdoctoral Associate at Indiana University (1969—70). This work represents generalizations of parts of the author's Ph. D. thesis which was directed by N. SUZUKI.

1. Structure theorems

In this section we prove the following main structure theorem.

Theorem 1. *Let A be an operator on a Hilbert space H and $p(z, \bar{z})$ be a non-commutative complex polynomial for which $p(A, A^*)$ is a compact operator. Then there exists a unique sequence of central projections $\{P_i\}_{i=0}^{n}$ ($n \leqq \infty$) in $R(A)$ so that*

$$A = A_0 \oplus \sum_{i=1}^{n} \oplus A_i,$$

where $A_0 \equiv AP_0 H$ satisfies $p(z, \bar{z})$ [2]), $A_i \equiv A|P_i H$ ($i \geqq 1$) are primary operators with $p(A_i, A_i^*)$ compact and non-zero, and $K = H \ominus P_0 H$ is separable.

We are interested in studying this theorem in the special cases where $p(z, \bar{z})$ is one of the following polynomials: 1) $p(z, \bar{z}) = z - \bar{z}$, 2) $p(z, \bar{z}) = z\bar{z} - \bar{z}z$), 3) $p(z, \bar{z}) = z\bar{z}z - \bar{z}z^2$, 4) $p(z, \bar{z}) = 1 - \bar{z}z$, and 5) $p(z, \bar{z}) = z - z\bar{z}z$. Case 1) has been studied by M. S. Brodskiĭ and M. S. Livšic [3], and N. Suzuki's original work also concerns it. The cases 2) and 3) have been studied by H. Behncke [1] and [2]; and case 3) by A. Brown [4]. Behncke obtained his structure by using C^*-algebra methods while Suzuki's original work is based on W^*-algebraic techniques. Case 4) has been studied by B. Sz.-Nagy and C. Foiaş if T is a contraction and by the author [8], where results analogous to Theorem 1 appear.

The proof of the theorem will be based on a proposition from the theory of von Neumann algebras. Let M be a von Neumann algebra and T be an operator in M. The *support* of T is the projection P on $\overline{T^*H}$ and $P \in M$. The *central support* of T is the smallest projection $F \in Z \equiv M \cap M'$ which majorizes P. If $\mathscr{J}$ is a family of operators in M we say that F is the central support of $\mathscr{J}$ if it is the smallest projection in Z which majorizes the support of each $T \in \mathscr{J}$. A non-zero projection $Q \in M$ is called *minimal* if it is an atom in the lattice of projections in M, that is, whenever R is a non-zero projection in M such that $R \leqq Q$, then $R = Q$.

Proposition 1. *Let M be a von Neumann algebra such that $M \cap \mathscr{C} \equiv \mathscr{C}_M$ has central support I.* [3]) *Then the lattice of projections in Z (the center of M) is atomic, that is, each non-zero $P \in Z$ majorizes a non-zero minimal projection $Q \in Z$.*

Proof. Let $0 \neq P \in Z$. If $PT = 0$ for each $T \in \mathscr{C}_M$, then $(I-P)T = T$ for each $T \in \mathscr{C}_M$. Hence $I-P$ would majorize the central support of $\mathscr{C}_M$ and $I \leqq I-P$ which implies that $P = 0$. Thus there is a $T \in \mathscr{C}_M$ such that $PT \neq 0$. Furthermore we may assume that $T = T^*$ and $PT = T$. By the spectral decomposition of the compact

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[2]) We say that the operator T satisfies $p(z, \bar{z})$ if $p(T, T^*) = 0$.

[3]) $\mathscr{C}$ is the two sided ideal of compact operators in H.

selfadjoint operator T, we may conclude that $E=PE\neq 0$, where E is the spectral projection on an eigenspace corresponding to a non-zero eigenvalue of T. E is finite dimensional since T is compact and the eigenvalue associated with E is non-zero. It is easy to show that $E\in M$ (Proposition 1 in [14]). Since E is a finite dimensional projection in M we may choose a projection $E_1\in M$ so that $0<E_1\leqq E$ and E_1 is a minimum non-zero projection in M (E_1 may be chosen so that $0\neq\dim(E_1H)=\min\{\dim(FH)\colon F\in M \text{ and } 0\neq F\leqq E\}$). If we let Q be the central support of $E_1\in M$, then we shall show that Q is a non-zero minimal projection in Z which is majorized by P. Since $P\geqq E_1$, it is clear that $P\geqq Q$. Let $R\in Z$ such that $R\leqq Q$. If $RE_1=0$, then $(Q-R)E_1=E_1$; hence $Q\leqq Q-R$, which implies that $R=0$. Since E_1 is a minimal projection in M, if $RE_1\neq 0$, then we have that $RE_1=E_1$. Because R is a central projection, we obtain the inequality $0\neq R\leqq Q\leqq R$, and hence $R=Q$. Therefore we have shown that Q is a minimal projection in Z.

Using this proposition we now prove Theorem 1.

Proof. First we describe the subspace $H\ominus P_0H$ which occurs in the statement of the theorem. Let $w(A,A^*)=\prod_{i=1}^{n}A^{k_i}A^{*m_i}$ be a word in A and A^*, that is, k_i and m_i are non-negative integers, possibly zero, and n is any positive integer. Denote by $\mathcal{M}$ the subspace of H generated by $\{w(A,A^*)x\colon x\in p(A,A^*)H$ and $w(A,A^*)$ is any word in A and $A^*\}$. The image of a compact operator is a separable subspace; hence $p(A,A^*)H$ is separable and thus the separability of $\mathcal{M}$ follows from the construction of $\mathcal{M}$. It is also clear that $\mathcal{M}$ is invariant under A and A^* and hence $\mathcal{M}$ reduces A, that is, if Q is the projection on $\mathcal{M}$, then $Q\in R(A)'$. Let T be an arbitrary operator in $R(A)'$ and $y\in\mathcal{M}$ be of the form $w(A,A^*)x$, where $x=p(A,A^*)z$. Then $Ty=Tw(A,A^*)x=w(A,A^*)Tx=w(A,A^*)p(A,A^*)Tz\in\mathcal{M}$; thus $\mathcal{M}$ is invariant under $T\in R(A)'$. Since $R(A)^*=R(A)$, we may conclude that $Q\in R(A)''=R(A)$ and therefore that $Q\in Z(A)\equiv R(A)\cap R(A)'$.

Denote by P_0 the central projection $I-Q$ and by A_0 the restriction of A to P_0. Next we shall show that $p(A_0,A_0^*)=0$. If $x\in H_0\equiv P_0H$, then $x=(I-Q)x$ and $p(A_0,A_0^*)x=Qp(A_0,A_0^*)x=Qp(A_0,A_0^*)(I-Q)x=Q(I-Q)p(A_0,A_0^*)x=0$. Furthermore, since Q is a central projection in $R(A)$, $\mathcal{M}$ is generated as before, with A replaced by AQ. If we denote by A_Q the operator $A|QH$, then $H\ominus H_0=\mathcal{M}$ is generated by words in A_Q and A_Q^* acting on $p(A_Q,A_Q^*)$.

The algebra $R(A)_Q=\{T|QH\colon T\in R(A)\}$ is equal to $R(A_Q)$ and $Z(A)_Q=Z(A_Q)$ [6]. By our remarks above, the identity operator on QH is the central support of the set of operators consisting of $p(A_Q,A_Q^*)$ multiplied by words in A_Q and A_Q^*. Each of these operators is compact and thus I_Q is the central support of $\mathscr{C}_{R(A_Q)}$. By Proposition 1, the lattice of projections in $Z(R(A_Q))$ is a complemented atomic lattice. By Zorn's lemma we may choose a maximal family $\{\tilde{P}_i\}_{i=1}^{n}$ ($n\leqq\infty$) of mutually orthogonal minimal projections in $Z(A_Q)$. This family is countable since

$QH=\mathcal{M}$ is separable and $\sum_{i=1}^{n} \tilde{P}_i = I_Q$ since the family is maximal. Because $Z(A_Q)=$ $=Z(A)_Q$ there are projections $\{Q_i\}_{i=1}^{n} \subset Z(A)$ such that $Q_i|QH=\tilde{P}_i$. If we define $P_i \equiv Q_i Q$, then $P_i|QH=\tilde{P}_i$ and $\{P_i\}_{i=1}^{n}$ is a family of mutually orthogonal minimal projections in $Z(A)$ with the property that $\sum_{i=1}^{n} P_i = Q$.

Since P_i is minimal projection in $Z(A)$, it follows that A_{P_i} is primary. Since $p(A_{P_i}, A^*_{P_i}) = p(A, A^*)|P_i H$, it is clear that $p(A_{P_i}, A^*_{P_i})$ is compact; however, we must show that $p(A_{P_i}, A^*_{P_i}) \neq 0$. If we assume that $p(A_{P_j}, A^*_{P_j})=0$ for some $j \geqq 1$, then $w(A_{P_j}, A^*_{P_j})p(A_{P_j}, A^*_{P_j})=0$, for any word $w(A_{P_j}, A^*_{P_j}) = \prod_{i=1}^{n} A^{k_i}_{P_j} A^{*m_i}_{P_j}$. We would then have that $\{0\} = w(A_{P_j}, A^*_{P_j})p(A_{P_j}, A^*_{P_j})P_j H = P_j(w(A, A^*)p(A, A^*)H$; thus it would follow that $P_j Q=0$. Therefore $P_j \perp Q$, which is a contradiction, since P_j is non-zero and $P_j \leqq Q$.

Remark 1. The argument given in the paragraph above is valid if P_j is any projection in $R(A)'$. That is, if N is a reducing space of A on which $p(A|N, A^*|N)=0$, then $N \subset P_0 H$.

Remark 2. The central support P of an operator $T \in M$ is also the central support of T^* and P also majorizes the projection on the smallest reducing space of T which contains TH. Thus we see that Q, as defined in the proof of Theorem 1, is the central support of $p(A, A^*)$.

For each $i \geqq 1$, we have that $0 \neq p(A_i, A_i^*)$ and thus the dimension of $p(A_i, A_i^*)H_i$ ($H_i \equiv P_i H$) is $\geqq 1$. Therefore, if $p(A, A^*)$ is itself of finite rank, then the decomposition given in Theorem 1 is finite.

Corollary 1. *Let A be an operator and $p(z, \bar{z})$ a nonmmutative polynomial for which $p(A, A^*)$ has finite rank. Then the decomposition in Theorem 1 is finite, that is, the index n in Theorem 1 is finite.*

Proof. The decomposition of A given by Theorem 1 has the property that $\dim(p(A, A^*)H) = \sum \dim(p(A_i, A_i^*)H_i)$ and for $i \geqq 0$ $\dim(p(A_i, A_i^*)H_i) \neq 0$.

In some cases we may wish to consider more than one non-commutative polynomial of z and $\bar{z}$. We can then extend the above idea so as to include this situation. For simplicity we shall only consider the case of two non-commutative polynomials.

Proposition 2. *Let A be an operator and $p(z, \bar{z})$ and $q(z, \bar{z})$ be commutative polynomials. Then there exists unique central projections, P_i $(i=1, 2, 3; 4)$ in $R(A)$ such that $A = A_1 \oplus A_2 \oplus A_3 \oplus A_4$, where $A_1 \equiv A|P_1 H$ satisfies p and q, A_2 satisfies p and has no reducing subspace on which it satisfies q, A_3 satisfies q and has no reducing subspace on which it satisfies p, and A_4 has no reducing space on which it satisfies either p or q.*

Proof. Let Q_1 be the central support of $p(A, A^*)$ and Q_2 the central support of $q(A, A^*)$. Let $Q_1 \cdot Q_2 = P_4$; then by Remark 1 $A_4 \equiv A|P_4 H$ has no reducing space on which A_4 satisfies either p or q. Let Q_3 be the central support of the set $\{p(A, A^*), q(A, A^*)\}$, that is $Q_3 = Q_2 + Q_1 - Q_1 Q_2$. If $P_1 = I - Q_3$, $P_2 = Q_3 - Q_1$, and $P_3 = Q_3 - Q_2$, then $\{P_1, P_2, P_3, P_4\}$ satisfy the conclusion of the proposition.

Remark. As a special case of Proposition 2 we may consider only one polynomial $p(z, \bar{z})$. In this case we decompose A into $A_0 \oplus A_1$ where $p(A_0, A_0^*) = 0$. By the remark following the proof of Theorem 1, we note that H_1, the space on which A_1 is defined, is generated by $\{w(A, A^*)p(A, A^*)H$: $w(A, A^*)$ is any word in A and $A^*\}$. This result is known in some special cases. Livšic and Brodskiĭ [3] call an operator simple if it has no reducing space on which it is selfadjoint. In this case H_1 is generated by $\{A^n(A - A^*)H$: $n = 0, 1, 2, \ldots\}$ and $A|H_1$ is called the simple part of A. Halmos [9] calls an operator abnormal if it has no reducing space on which it is normal. Finally B. Sz.-Nagy and C. Foiaş use the terminology completely non-unitary for contractions with no reducing spaces on which they are unitary. This latter notation seems the most descriptive of the situation.

If we combine Theorem 1 and Proposition 2 we obtain the form that the structure theorem takes in many of its applications.

Theorem 2. *Let A be an operator and $p(z, \bar{z})$ and $q(z, \bar{z})$ be two non-commutative polynomials such that $p(A, A^*)$ is compact. Then there exists unique central projections $\{P_i\}_{i=1}^n$ $(n \leqq \infty)$ in $R(A)$ so that*

$$A = A_1 \oplus A_2 \oplus A_3 \oplus \sum_{i \geqq 4} \oplus A_i,$$

where $A_i \equiv A|P_i H$, $p(A_1, A_1^*) = q(A_1, A_1^*) = 0$, $p(A_2, A_2^*) = 0$, A_2 has no reducing space on which $q(A_2, A_2^*) = 0$, $q(A_3, A_3^*) = 0$, and A_3 has no reducing subspaces on which $p(A_3, A_3^*) = 0$, A_i $(i \geqq 4)$ are primary operators with $p(A_i, A_i^*)$ compact, and each A_i $(i \geqq 4)$ has no reducing subspace on which $q(A_i, A_i^*) = 0$ or $p(A_i, A_i^*) = 0$.

Proof. From Proposition 2 we obtain the projections P_1, P_2, and P_3. Applying Theorem 1 to the operator A_{P_4} and the algebra $R(A_{P_4})$ we complete the decomposition of A.

We now turn to the structure of primary operators A for which $p(A, A^*)$ is compact and non-zero. Here the algebraic character of the operator plays the important role. This fact was first noticed by Suzuki for primary operators with compact imaginary parts. The following proposition is essentially a restatement of Proposition 2 in [14]. Let A be a primary operator and $p(z, \bar{z})$ a non-commutative polynomial for which $p(A, A^*)$ is compact and non-zero. The projections on proper subspaces of $\mathrm{Re}\,(p(A, A^*))$ and $\mathrm{Im}\,(p(A, A^*))$ corresponding to non-zero proper values have finite rank and belong to $R(A)$. Since at least one such projection exists and is non-zero, we have that $R(A)$ contains finite dimensional projections and hence $R(A)$ contains minimal projections. Therefore the von Neumann algebra $R(A)$ is a factor of type I and the dimension n of a minimal projection in $R(A)$ is uniquely determined. The number n is a unitary invariant for A and is called the multiplicity of A.

Proposition 3. *Let A be a primary operator and $p(z, \bar{z})$ a non-commutative polynomial for which $p(A, A^*)$ is compact and non-zero. If n is the multiplicity of A, then $R(A)'$ (the commutant of $R(A)$) is of type I_n.*

The proof is similar to the proof of Proposition 2 in [14].

The type of algebra generated by an operator has been studied by many authors. As a corollary to Proposition 3, we have the following result.

Proposition 4. *If A is an operator and $p(z, \bar{z})$ is a non-commutative polynomial for which $p(A, A^*)$ is compact, then $R(A)$ is a type I algebra if and only if A_0 (given by Theorem 1) generates an algebra of type I.*

Now for special cases we can determine certain operators that generate type I algebras.

Corollary 2. *Let A be an operator for which $p(A, A^*)$ is compact. Then A generates a type I von Neumann algebra if*

i) $p(z, \bar{z}) = z - \bar{z}$, ii) $p(z, \bar{z}) = \bar{z}z - z\bar{z}$, *or* iii) $p(z, \bar{z}) = 1 - z\bar{z}$.

Proof. This result is known for case i) (Suzuki [14]) and case ii) (Behncke [1]). Case iii) follows since an isometry generates a type I von Neumann algebra.

Remark. Carl Pearcy gives examples of partial isometric operators which do not generate type I von Neumann algebras [10]. Hence for $p(z, \bar{z}) = z - z\bar{z}z$ and an operator A such that $p(A, A^*)$ is compact, the algebra $R(A)$ need not be type I.

Now we complete the algebraic structure of operators A for which $p(A, A^*)$ is compact and non-zero. We shall show that when the operator A is also primary, then it is just the direct sum of n copies of an irreducible operator V with the properties that $p(V, V^*)$ is compact and non-zero. The following theorem is similar to Theorem 3 in [14] where the case $p(z, \bar{z}) = z - \bar{z}$ was considered.

Theorem 3. *Let A be a primary operator and $p(z, \bar{z})$ a non-commutative polynomial such that $p(A, A^*)$ is compact and non-zero. If m is the multiplicity of A, then A is unitarily equivalent to an operator $V \otimes I_m$, where V is an irreducible operator with $p(V, V^*)$ compact and non-zero and I_m is the identity operator on an m-dimensional Hilbert space.*

Proof. A von Neumann algebra of type $I_{\alpha m}$ is spatially isomorphic to $\mathcal{L}(K)\otimes\otimes\{\lambda I_m\}$, where $\mathcal{L}(K)$ is the algebra of all bounded operators on an α-dimensional Hilbert space K and λI_m are the scalar multiples of the identity operator I_m on an m-dimensional Hilbert space [6]. Thus A is unitarily equivalent to an operator of the form $V\otimes I_m \in \mathcal{L}(K)\otimes\{\lambda I_m\}$. One can then show that V must be irreducible.

If $p(z, \bar{z})$ is a non-commutative polynomial, we say that the operator A has *p-rank* r if rank $p(A, A^*)$ is r. Using strictly algebraic ideas we obtain the following two corollaries of Theorem 3.

Corollary 3. *If A is a primary operator with p-rank r and multiplicity m then A is unitarily equivalent to $V\otimes I_m$ and the p-rank of V is n where $r = n\cdot m$.*

Corollary 4. *Let A be a primary operator with p-rank r. If the multiplicity of A is 1 and r is a prime number, then A is either irreducible or else A is unitarily equivalent to $V\otimes I_r$, in which case the p-rank of V is 1.*

We wish to illustrate this theory with examples using the specific non-commutative polynomials mentioned at the beginning of this section. Operators A with $A-A^*$ compact have been extensively studied by various authors; see [3] and [14]. In this case A is uniquely decomposed by central projections in $R(A)$ as

$$A = A_0\oplus \sum_{i=1}^{n}\oplus A_i \qquad (n \leqq \infty),$$

where A_0 is a self adjoint operator and each A_i $(i\geqq 1)$ is a primary operator with $\operatorname{Im} A_i$ compact. By theorem 3 each $A_i = V_i\otimes I_{n_i}$, V_i is irreducible with $\operatorname{Im} V_i$ compact and non-zero, and $n_i<\infty$. These results are due to N. Suzuki [14].

Following Suzuki's original work, H. Behncke [1] used the theory of C^*-algebras to prove the analogous theorem when $p(z) = \bar{z}z - z\bar{z}$. If A^*A-AA^* is compact then A is uniquely decomposed by central projections in $R(A)$ as

$$A_0\oplus \sum_{i=1}^{n}\oplus A_i \qquad (n \leqq \infty),$$

where A_0 is normal, each A_i is primary with $A_i^*A_i - A_iA_i^*$ compact and each $A_i = V_i\otimes I_{n_i}$, where V_i is irreducible $V_i^*V_i - V_iV_i^*$ is compact and non-zero, and $n_i<\infty$.

Using the polynomial $p(z, \bar{z}) = z\bar{z}z-\bar{z}z^2$ and $q(z, \bar{z}) = \bar{z}z - z\bar{z}$ and Theorem 2, we can obtain the decomposition given by H. Behncke in [2] whenever $p(A, A^*)$ is compact.

For contraction operators A with $p(A, A^*) = I-A^*A$ compact the algebraic decomposition has been given by the author [8]. If we consider the polynomials $p(z, \bar{z}) = 1-\bar{z}z$ and $q(z, \bar{z}) = 1-z\bar{z}$ and an operator A for which $p(A, A^*)$ is compact, then Theorems 2 and 3 give the following unique central decomposition.

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \sum_{i=3}^{n} \oplus A_i \qquad (n \leqq \infty),$$

where A_0 is unitary, A_1 is a forward unilateral shift, A_2 is a backward unilateral shift and each A_1 ($i \geqq 3$) is a primary operator. Furthermore, for $i \geqq 3$, $A_i = V_i \otimes I_{n_i}$, where V_i is irreducible, $I - V_i^* V_i$ is compact and non-zero, $n_i < \infty$, and V_i is completely non-isometric.

2. Applications

In this section we give an application of Theorem 1 based on the theory of spectral operators [7]. The results of this section give striking examples of how the algebraic decomposition of an operator can be used to determine its exact structure.

J. Schwartz [12] and N. Suzuki [15] have determined a structure theorem for the spectral operator A whenever $A - A^*$ is compact. We will give the analogous result whenever: i) $A^*A - AA^*$, ii) $I - A^*A$, or iii) $AA^*A - A^*A^2$ is compact. This will correspond to three specific uses of Theorem 1.

In what follows we shall use several results concerning the Calkin algebra associated with $\mathscr{L}(H)$. The algebra $\mathscr{L}(H)/\mathscr{C}$ ($\mathscr{C}$ is the compact operators in $\mathscr{L}(H)$) is a B^* algebra with involution * and it is called the Calkin algebra associated with H. If $\hat{A}$ denotes the image of A in $\mathscr{L}(H)/\mathscr{C}$, then $(\hat{A})^* = \hat{A}^*$ and $\sigma(\hat{A}) \subset \sigma(A)$. For details concerning this algebra we refer to [5].

The following lemma gives conditions on a spectral operator A which imply that the quasinilpotent part is compact or equivalently, that the operator $\hat{A}$ is a scalar type operator in $\mathscr{L}(H)/\mathscr{C}$.

Lemma 1. *Let A be a spectral operator with the canonical decomposition $A = S + N$, where S is a scalar type operator and N is a quasinilpotent operator. Then N is compact if any of the following operators* i) $A^*A - AA^*$, ii) $A^* - A$, iii) $I - A^*A$, *or* iv) $AA^*A - A^*A^2$ *is compact.*

Proof. Since $A = S + N$, we have $\hat{A} = \hat{S} + \hat{N}$ as the canonical decomposition of $\hat{A}$ in $\mathscr{L}(H)/\mathscr{C}$. In cases i) and ii) we clearly have that $\hat{A}$ is normal. Since the decomposition into scalar and quasinilpotent parts is unique, we may conclude that $\hat{N} = 0$ and therefore N is compact. Part i) was proven by Schwartz in [12].

In the case iii), $\hat{A}$ is an isometry. It can be shown directly that isometric spectral operators are normal.

In case iv) we are considering an operator $\hat{A} = B$ such that $BB^*B - B^*B^2 = 0$. A. Brown [4] has completely characterized these operators; he shows that $B = VD$, where V is an isometry, $D \geqq 0$ and $VD = DV$. Again one can directly show that a spectral operator B satisfying iv) is normal. However in case iii) and iv) the operator $\hat{A}$ is also subnormal.

J. STAMPFLI has shown [13] that in a separable Hilbert space every subnormal spectral operator is normal. His proof is independent of separability and hence can be used here. Hence in either iii) or iv) we may conclude that $\hat{N} = 0$ and therefore N is compact.

Now we present the main theorem of this section.

Theorem 4. *Let A be a spectral operator on a Hilbert space H. Whenever at least one of the operators* i) $A^*A - AA^*$, ii) $A^* - A$, iii) $I - A^*A$, *or* iv) $AA^*A - A^*A^2$ *is compact, then A decomposes into the algebraic direct sum*

$$A = A_0 \dotplus \sum_{i=1}^{n} \dotplus (\lambda_i I_i + N_i) \quad (n \leqq \infty) \quad on \quad H = H_0 \dotplus \sum_{i=1}^{n} \dotplus H_i;$$

where $\{H_i\}_{i=0}$ are invariant subspaces for A, $A_0 \equiv A|H_0$ is scalar, I_i is the identity operator on H_i, $(\lambda_i I_i + N_i) \equiv A|H_i$, $\lambda_i \in \sigma(A)$, N_i is a compact quasinilpotent operator and $\|N_i\| \to 0$ if $n = \infty$. Furthermore in the cases ii) *and* iii) *we also have, that respectively, A_0 is similar to a self-adjoint operator with* $\operatorname{Im} \lambda_i \to 0$ *if $n = \infty$; and A_0 is similar to a unitary operator with $|\lambda_i| \to 1$ if $n = \infty$. Finally, the non-scalar summand $\sum_{i=1}^{n} \dotplus H_i$ is separable.*

Proof. Let A be a spectral operator with canonical decomposition $A = S + N$ where N is compact. Let R be the invertible operator for which RSR^{-1} is normal and let $B \equiv RAR^{-1}$, $T \equiv RSR^{-1}$, and $L \equiv RNR^{-1}$. Now L is also compact and T is normal, so that $\hat{B} = \hat{T}$ and $B^*B - BB^*$ is compact.

Using the polynomial $p(z, \bar{z}) = \bar{z}z - z\bar{z}$ in Theorem 1, the operator B decomposes as

$$B = B_0 \oplus \sum_{i=1}^{n} \oplus B_i \quad (n \leqq \infty), \quad \text{with} \quad H = H_0 \oplus \sum_{i=1}^{n} \oplus H_i,$$

where $B_0 \equiv B|H_0$ is normal and $B_i \equiv B|H_i$ is a primary operator ($i \geqq 1$).

Each B_i ($i \geqq 1$) is also a spectral operator and has the canonical decomposition $B_i = T_i + L_i$. Since T, $L \in R(B)'$ and the decomposition of B was by central projections in $R(B)$, the operator T_i is $T|H_i$ and L_i is $L|H_i$. Each T_i is normal and belongs to the center of the algebra $R(B_i)$. Since B_i is a primary operator, we may conclude that $T_i = \lambda_i I_i$ for some scalar λ_i (I_i is the identity operator on H_i). Because $\{\lambda_i\} = \sigma(T_i) \subset \sigma(T) = \sigma(B) = \sigma(A)$, we note that $\lambda_i \in \sigma(A)$. Therefore B is decomposed as $B = B_0 \oplus \sum_{i=1}^{n} \oplus (\lambda_i I_i + L_i)$ $(n \leqq \infty)$; furthermore, since L is compact, $\|L_i\| \to 0$ if $n = \infty$.

If A satisfies any of the conditions i)—iv), we have by Lemma 1 that N is compact. Therefore, A has the decomposition given above. Now we shall discuss the special cases ii) and iii). In either case $\sigma(B)=\sigma(A)\supset\sigma(B_0)$ and $\sigma(\hat{A})=\sigma(\hat{B})\supset\sigma(\hat{B}_0)$. In case ii), $\sigma(\hat{A})$ is real and hence B_0 is a normal operator with $\sigma(\hat{B}_0)$ real, that is, $\hat{B}_0$ is self adjoint and Im (B_0) is compact. By reordering, in the above decomposition, and redenoting B_0 as the selfadjoint part of B_0, we obtain in case ii):

$$B = B_0 \oplus \sum_{i=1}^{n} \oplus (\lambda_i I_i + L_i) \qquad (n \leqq \infty),$$

where B_0 is selfadjoint, Im $\lambda_i \to 0$ and $\|L_i\| \to 0$ if $n=\infty$. In case a particular λ_i arises from the previous B_0 we simply define $L_i \equiv 0$. Now if we premultiply by R and postmultiply by R^{-1} we obtain the desired result

$$A = A_0 \dot{+} \sum_{i=1}^{n} \dot{+} (\lambda_i I_i + N_i) \quad (n \leqq \infty) \quad \text{on} \quad H = H_0 \dot{+} \sum_{i=1}^{n} \dot{+} H_i,$$

where A_0 is a scalar operator with real spectrum, Im $\lambda_i \to 0$ and $\|N_i\| \to 0$ if $n=\infty$.

In case iii) we may proceed as in case ii). Since spectral isometries are unitary, it follows that $\hat{A}$ is unitary; thus $\sigma(\hat{A})\subset\{z:|z|=1\}$ and $\sigma(\hat{B})\subset\{z:|z|=1\}$. Thus B_0 is a normal operator with $\sigma(\hat{B}_0)$ on the boundary of the unit disk. Hence $B_0 = U \oplus \sum \oplus \lambda_i I_i$, where U is a unitary operator, $\{\lambda_i\}=\sigma(B_0)\setminus\{z:|z|=1\}$, and I_i is the identity operator on the eigenspace corresponding to λ_i. We may redenote B_0 as the unitary part of B_0 and obtain the decomposition:

$$B = B_0 \oplus \sum_{i=1}^{n} \oplus (\lambda_i I_i + L_i) \qquad (n \leqq \infty),$$

where B_0 is unitary and $\|L_i\| \to 0$ if $n=\infty$. The set $\{\lambda_i\}$ does not have limit points in the set $\{z:|z|<1\}$, since $\partial\sigma(A)\subset\sigma(\hat{A})\cup$ {isolated eigenvalues of A of finite multiplicity} [11]; therefore, we conclude that $|\lambda_i| \to 1$ if $n=\infty$.

By premultiplying by R and postmultiplying by R^{-1} we finally obtain that:

$$A = A_0 \dot{+} \sum_{i=1}^{n} \dot{+} (\lambda_i I_i + N_i) \quad (n \leqq \infty) \quad \text{on} \quad H = H_0 \dot{+} \sum_{i=1}^{n} \dot{+} (\lambda_i I_i + N_i),$$

where A_0 is a scalar type operator with $\sigma(A_0)$ lying on the circumference of the unit circle, $|\lambda_i| \to 1$ and $\|N_i\| \to 0$ if $n=\infty$.

Remark. The use here of Theorem 1 is similar to that made by N. Suzuki in the case ii) [15]. However, the use of the spectral properties of an operator A and $\hat{A}$ are details that differ from the proof of ii) in [15]. This Theorem for case ii) was originally given by J. Schwartz using completely different methods.

Remark. By the argument given in the first part of the proof, we see that the decomposition in the theorem holds for any spectral operator with compact quasinilpotent part.

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