Weighted bilateral shifts of class C_{01}

By FRANK GILFEATHER in Honolulu (Hawaii, U.S.A.) [1])

In this paper all operators are bounded operators on separable Hilbert spaces. B. SZ.-NAGY and C. FOIAŞ have developed a classification theory for contraction operators ($\|T\| \leqq 1$) which is based on the asymptotic behavior of the operator and its adjoint [6; Chapter II, Section 4]. A contraction operator T on H is called type C_{01} if $T^n h \to 0$ for all $h \in H$ and $T^{*n} h \not\to 0$ for each $h \in H$, $h \neq 0$. For complete details of this classification theory we refer the reader to [6], Chapter II, Section 4.

Some properties of the operators in C_{01} are known. Whenever $T \in C_{01}$ and the rank of $I - T^*T$ is finite, then the rank of $I - TT^*$ is *strictly* smaller than the rank of $I - T^*T$; *cf.* [6], Proposition I. 2. 1 and Theorems II. 1. 1—2. Hence it follows from [6], Theorem VI. 4. 1, that $\sigma_p(T)$ includes the whole open unit disk D.

A contraction T is called a *weak contraction* if $I - T^*T$ is of trace class and if $\sigma(T) \neq \bar{D}$. In [6], Chapter VIII, the structure of weak contractions is extensively developed. Our examples shall show that this structure cannot be extended to the Schatten class $\mathfrak{S}_p$ for any $p > 1$; *cf.* [1], X. 1. 9.

In this note we present examples of contraction operators in the class C_{01} which have no point spectrum. Example 1 will show that the spectrum can lie on the circumference of the unit disk and the point spectrum can be empty even when $I - T^*T$ is an $\mathfrak{S}_p$ operator with $p > 1$. Furthermore the example will give realizations of C_{01} operators for which T has a cyclic vector. Examples will be in C_{01} with $\sigma(T) = \bar{D}$. Specifically all the examples will have in common the following properties:

(i) T is irreducible,

(ii) $\sigma_p(T^*) = \sigma_p(T) = \emptyset$,

(iii) T has a cyclic vector,

(iv) T^* has *no* invariant subspaces on which it is an isometry.

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[1]) This work was done while the author was an Office of Naval Research Postdoctoral Associate at Indiana University. The author acknowleges that these results are examples for questions raised by C. FOIAŞ.

The examples will be generated by weighted bilateral shifts. Let H be a separable Hilbert space and $\{e_n\}$ $(n=0, \pm 1, \pm 2, \ldots)$ an orthonormal basis. Let T be the operator which maps e_n onto $\omega_n e_{n+1}$ $(n=0, \pm 1, \pm 2, \ldots)$, where ω_n is a complex number. The set $\{\omega_n\}$ is called the *weights* of T. T is a contraction iff $|\omega_n| \leqq 1$ for every n. The following proposition determines the class to which T belongs.

Proposition. *Let T be a weighted bilateral shift with weights $\{\omega_n\}$ such that T is a contraction.*

a) $T \in C_{0\cdot}$ *if and only if either* (i) *for every positive integer N there exists an $n > N$ such that $\omega_n = 0$, or* (ii) *for some subsequence $\{n_i\}$ of positive integers with $\omega_{n_i} \neq 0$ the infinite product $\prod |\omega_{n_i}|$ diverges.*

b) $T \in C_{1\cdot}$ *if and only if each $\omega_i \neq 0$ and the infinite product $\prod_{i \geqq 0} |\omega_i|$ converges.*

The proof of this proposition is straightforward and appears in [2], Chapter II. As a corollary of this result we determine when T is a C_{01} contraction.

Corollary. *Let T be a weighted bilateral shift with weights $\{\omega_n\}$ such that T is a contraction. Then $T \in C_{01}$ if and only if, for all $n = 0, \pm 1, \ldots,$*

(i) $\prod_{i \geqq n} |\omega_i|$ *diverges, and* (ii) $\prod_{i \leqq n} |\omega_i|$ *converges.*

Remark. If we assume that $\omega_i \neq 0$ for all i, then $T \in C_{01}$ if and only if $\prod_{i \leqq 0} |\omega_i|$ converges and $\prod_{i \geqq 0} |\omega_i|$ diverges.

Now we shall present the first example.

Example 1. *Let T be the weighted bilateral shift with weights*

$$\omega_n \equiv \begin{cases} \left(\dfrac{n-1}{n}\right)^{\frac{1}{2}} & \text{if} \quad n > 1, \\[2ex] \dfrac{n^2-1}{n^2} & \text{if} \quad n < -1, \quad \text{and} \\[2ex] 1 & \text{otherwise.} \end{cases}$$

The operator T is in the class C_{01}, has properties (i)—(iv) *and furthermore $I - T^* T$ is an $\mathfrak{S}_p$ operator for $p > 1$.*

First we shall show that $T \in C_{01}$. Since all the weights are less than or equal to 1 we conclude that $\|T\| \leqq 1$. The infinite product $\prod \left(\dfrac{n-1}{n}\right)^{\frac{1}{2}}$ has its partial products converging to zero. By the proposition we can conclude that $T\in C_{0\cdot}$. The series $\sum\frac{1}{n^2}$ is convergent and hence the infinite product $\prod\frac{n^2-1}{n^2}$ does convergence. From our corollary and the remark following it, we have $T\in C_{01}$. Furthermore the products $\beta_k=\prod_{i\leq k}\omega_i$ are convergent and have the property that $\beta_k\to 0$ as $k\to+\infty$.

Now we shall discuss the properties (i)—(iv). Properties (i) and (iv) are easily shown. That T is irreducible can be deduced from a result due to R. L. KELLEY [3], Problem 129. Assume that T^* has an invariant subspace on which it is an isometry and h is any non-zero vector in that subspace. Since $\{e_n\}$ is an orthonormal basis, we have, $h=\sum_{k=-\infty}^{\infty}\alpha_k e_k$ and $T^{*n}h=\sum_{k=-\infty}^{\infty}\left(\prod_{i=0}^{n-1}\omega_{i-n}\right)e_{k-n}$. For n large enough ($n\geq 4$) and for some k with $\alpha_k\neq 0$, we will have $\left|\prod_{i=0}^{n}\omega_{i-n}\right|\neq 1$. When this happens, then $\|T^{*n}h\|\neq\|h\|$. Thus we reach a contradiction to our assumption that T^* had an invariant subspace on which T^* was an isometry. For $p>1$ the sum $\sum_{-\infty}^{\infty}(1-\omega_i^2)^p$ is just the sum $\sum_{-\infty}^{\infty}\|(I-T^*T)e_i\|^p$. By our choice of ω_i, this sum is finite whenever $p>1$, and hence T belongs to the Schatten class $\mathfrak{S}_p$.

The convergence properties of the weights will enable us to show property (ii). As we mentioned in the introduction, of most interest is the property that $\sigma_p(T)=\emptyset$. It follows from [5], Theorem 5, that $\sigma(T)=\{\lambda:|\lambda|=1\}$. Therefore since T is a completely non-unitary contraction, we have $\sigma_p(T)=\sigma_p(T^*)=\emptyset$. However this is easy to see by directly calculating the spectral radius of T^{-1}. From our definition of T it follows that $\|T^{-n}\|\leq n$ $(n>1)$ and hence the spectral radius of T^{-1} is 1. Since the spectral radius of T and T^{-1} is 1 we must have that $\sigma(T)\subset\{\lambda:|\lambda|=1\}$.

In order to show (iii) we shall construct the cyclic vector using the criterion for a cyclic vector of the simple bilateral shift (that is, all weights are 1 and the multiplicity is 1) [4], p. 114. In order to do this we first show that the simple bilateral shift is quasi affine to T. We have already mentioned that $\beta_n=\prod_{i\leq n}\omega_i$ is defined for all n. If we define X to be the operator which maps e_n to $\beta_n e_n$, then X is an injective selfadjoint operator on H. For each vector e_n we have $TXe_n=T\beta_n e_n=$ $=\omega_n\beta_n e_{n+1}=\beta_{n+1}e_{n+1}=Xe_{n+1}=XSe_n$, where S is the simple bilateral shift. Let f be a cyclic vector for S. Thus $\operatorname{span}\{T^nXf\}=\operatorname{span}\{XS^nf\}=X\operatorname{span}\{S^nf\}=H$ and Xf is a cyclic vector for T.

If we choose different weights we can construct an example of a C_{01} operator with properties (i)—(iv) and with the additional property that $\sigma(T)=\{\lambda:|\lambda|\leq 1\}$.

Example 2. *Let T be the weighted bilateral shift with weights*

$$\omega_n \equiv \begin{cases} \frac{n^2-1}{n^2} & \text{if } n<-1, \\ \frac{1}{k} & \text{if } n=k^3,\ k>1, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Then T belongs to C_{01}, has properties (i)—(iv) *and the property that $\sigma(T)$ is the closed unit disk.*

That T is in C_{01} and satisfies (i) and (iv) is clear. To see that T has a cyclic vector we proceed exactly as in the proof of Example 1. R. L. Kelley has shown that $\sigma(T)$ is connected [5], p. 354. Since $\sigma(T)$ has circular symmetry [3], p. 75, we have that $\sigma(T)=\{\lambda:|\lambda|\leqq 1\}$. To show property (ii) let us assume that $\lambda\in\sigma_p(T)$. T is completely non-unitary, hence $|\lambda|\neq 1$. Since all the weights are non-zero we also know that $0\notin\sigma_p(T)$. Let $h=\Sigma\alpha_n e_n$ be an eigenvector for eigenvalue λ of T. By matching the corresponding Fourier coefficients of Th and λh, we obtain for all n

$$(*)\qquad \omega_{n-1}\alpha_{n-1}=\lambda\alpha_n.$$

If $\alpha_0=0$, then $h=0$ since our weights are all non-zero. For $n>0$ we obtain from $(*)$ that

$$\alpha_n=\lambda^{-n}\left(\frac{\beta_{n-1}}{\beta_0}\right)\alpha_0.$$

If we let $n+1=k^3$, then

$$\alpha_{n+1}=\lambda^{-k^3}(k!)^{-1}\beta_0^{-1}\alpha_0.$$

This sequence does *not* converge to zero whenever $|\lambda|<1$. Hence $\{\alpha_n\}$ *cannot* be the Fourier coefficients of a vector $h\in H$. By this contradiction we conclude that $\sigma_p(T)=\emptyset$.

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