Operators unitary in an indefinite metric and linear fractional transformations

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Introduction

There is a close connection [2] between unitary operators on a Hilbert space with an indefinite metric and linear fractional transformations defined on the unit ball of a certain operator algebra (general symplectic maps). Invariant subspace problems for indefinite metric-unitary operators are equivalent to fixed point problems for general symplectic maps. In this note we define three natural classes of general symplectic maps — elliptic, hyperbolic, and parabolic. A linear fractional transformation of the disk onto itself in the complex plane is elliptic if and only if it has a fixed point in the interior of the disk. We prove that this is true for general symplectic maps. We also prove a basic inequality (6). We illustrate the strength of these two fundamental facts by giving a new proof of a generalized version [1] of NAĬMARK's Theorem [3] that every commuting family of unitary operators on a Pontryagin space has an invariant maximal positive subspace.

Background

The notation to be used in this paper is the same as the notation in [1]. We describe it briefly in this section.

The bilinear form $Q(\ ,\ )$ on a complex Hilbert space H is called an indefinite inner product on H provided that H is the direct sum of two orthogonal subspaces H_+, H_- with respect to which $Q(\ ,\ )$ has the representation

$$Q(x, y) = (E_+ x, y) - (E_- x, y) \tag{1}$$

where $E_\pm$ are the orthogonal projections of H onto $H_\pm$, and x, y are two vectors in H. A closed subspace P of H which contains only vectors p for which $Q(p, p) \geqq 0$

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is called *positive*. A *maximal* positive subspace is positive and not properly contained in any positive subspace of H. If S is a subspace we let $S'=\{q|\,Q(s,q)=0 \text{ if } s\in S\}$. An operator U on H which satisfies $Q(Ux, Uy)=Q(x,y)$ for all x, y in H is called *Q-unitary*. Let $\mathbf{B}$ denote the set of operators from H_+ into H_- with norm $\leqq 1$.

The following facts are well known [1] [4] [5]. There is a natural one-one correspondence between maximal positive subspaces P of H and operators J in $\mathbf{B}$ such that $P=(I+J)H_+$; we write $P\sim J$.

If U is a Q-unitary operator, then the matrix for U with respect to H_+, H_- has the form

$$U=\begin{pmatrix}(1-J^*J)^{-\frac{1}{2}}\psi & J*(1-JJ^*)^{-\frac{1}{2}}\varphi\\ J(1-J^*J)^{-\frac{1}{2}}\psi & (1-JJ^*)^{-\frac{1}{2}}\varphi\end{pmatrix}. \tag{2}$$

where ψ and φ are unitary operators on H_+ and H_- respectively and J is an operator from H_+ to H_- with norm $\leqq 1$. The map $\mathfrak{F}$: $\mathbf{B}\to\mathbf{B}$ defined by

$$\mathfrak{F}(K)=(1-JJ^*)^{-\frac{1}{2}}[J\psi+\varphi K][\psi+J^*\varphi K]^{-1}(1-J^*J)^{\frac{1}{2}} \tag{3}$$

for all $K\in\mathbf{B}$ has the property

$$\text{if} \quad P\sim K, \quad \text{then} \quad UP\sim\mathfrak{F}(K). \tag{4}$$

If U is any Q-unitary operator with this property we write $U\sim\mathfrak{F}$ and say that U corresponds to $\mathfrak{F}$. If $U\sim\mathfrak{F}$ and $V\sim\mathfrak{F}$, then V is a scalar multiple of U. Any map $\mathfrak{F}$ that arises from a Q-unitary operator in the manner described above is called general symplectic. The set of all general symplectic maps is a group under composition and is denoted by $\mathscr{G}_1$. Note that if $\mathfrak{F}$ is defined by equation (3), then $\mathfrak{F}(0)=J$.

A simple inequality

Suppose that $K\in\mathbf{B}$ with $\|K\|<1$, suppose that $\mathfrak{F}\in\mathscr{G}_1$ and set $J=\mathfrak{F}(0)$. The elementary identity $J=(1-JJ^*)^{-\frac{1}{2}}J(1-J^*J)^{\frac{1}{2}}$ combined with the definition (3) of $\mathfrak{F}$ yields

$$\begin{aligned}\mathfrak{F}(K)-J&=(1-JJ^*)^{-\frac{1}{2}}\{[J\psi+\varphi K][\psi+J^*\varphi K]^{-1}-J\}(1-J^*J)^{\frac{1}{2}}\\ &=(1-JJ^*)^{-\frac{1}{2}}\{J\psi+\varphi K-J\psi-JJ^*\varphi K\}[\psi+J^*\varphi K]^{-1}(1-J^*J)^{\frac{1}{2}}\end{aligned}$$

and hence

$$\mathfrak{F}(K)-\mathfrak{F}(0)=(1-JJ*)^{\frac{1}{2}}\varphi K[\psi+J*\varphi K]^{-1}(1-J*J)^{\frac{1}{2}}. \tag{5}$$

Since $\|K\|<1$ the inequality $\|[\psi+J^*\varphi K^{-1}]\|\leqq\{1-\|K\|\}^{-1}$ is valid and equation (5) implies that

$$\begin{aligned}\|\mathfrak{F}(K)x-\mathfrak{F}(0)x\|&\leqq\|(1-JJ^*)^{\frac{1}{2}}\|\,\|K\|\{1-\|K\|\}^{-1}\|(1-J^*J)^{\frac{1}{2}}x\|\\ &\leqq\sqrt{2}\,\|K\|\{1-\|K\|\}^{-1}\{\|x\|^2-\|\mathfrak{F}(0)x\|^2\}^{\frac{1}{2}}.\end{aligned}$$

Now we extend this to a more general inequality. Suppose that $M \in \mathbf{B}$ and $\|M\| < 1$. There is a map $\mathfrak{G} \in \mathcal{G}_1$ such that $\mathfrak{G}(0) = M$ (c.f. Lemma 1. 1 [6]) and it is easy to see that $\|\mathfrak{G}^{-1}(K)\| < 1$ since $\|K\| < 1$. Since $\mathcal{G}_1$ is a group, $\mathfrak{F} \circ \mathfrak{G} \in \mathcal{G}_1$; thus if we substitute $\mathfrak{F} \circ \mathfrak{G}$ for $\mathfrak{F}$ and $\mathfrak{G}^{-1}(K)$ for K into the above inequality we get

$$\|\mathfrak{F}(K)x - \mathfrak{F}(M)x\| \leqq \sqrt{2}\, \|\mathfrak{G}^{-1}(K)\| \{1 - \|\mathfrak{G}^{-1}(K)\|\}^{-1} \{\|x\|^2 - \|\mathfrak{F}(M)x\|^2\}^{\frac{1}{2}}.$$

In other words

(6) $$\|\mathfrak{F}(K)x - \mathfrak{F}(M)x\| \leqq c\{\|x\|^2 - \|\mathfrak{F}(M)x\|^2\}^{\frac{1}{2}}$$

where c is a constant independent of $\mathfrak{F}$ and of x.

Three classes of maps in $\mathcal{G}_1$

Let $\mathfrak{F}^N$ ($\mathfrak{F}^{-N}$) denote the N^{th} iterate of the map $\mathfrak{F}$ ($\mathfrak{F}^{-1}$) in $\mathcal{G}_1$, for $N = 0, 1, 2, \ldots$. The set $\mathbf{B}^0 = \{M \in \mathbf{B}: \|M\| < 1\}$ is called the interior of $\mathbf{B}$.

Definition. Suppose that $\mathfrak{F}$ is in $\mathcal{G}_1$. An operator M in $\mathbf{B}^0$ will be called a uniformly elliptic [E], a uniformly parabolic [P], or a uniformly hyperbolic [H] point for $\mathfrak{F}$ provided that

[E] there is a number $\alpha < 1$ such that $\|\mathfrak{F}^{\pm N}(M)\| < \alpha$ for all N.

[P] $\mathfrak{F}^{\pm N}(M)$ is invertible for large N, $\|[\mathfrak{F}^{\pm N}(M)]^{-1}\| \to 1$, and $\|\mathfrak{F}^N(M) - \mathfrak{F}^{-N}(M)\| \to 0$.

[H] $\mathfrak{F}^{\pm N}(M)$ is invertible for large N, $\|[\mathfrak{F}^{\pm N}(M)]^{-1}\| \to 1$, and there is a $\delta > 0$, so that $\|[\mathfrak{F}^N(M) - \mathfrak{F}^{-N}(M)]x\| \geqq \delta\|x\|$ for all N and all $x \in H_+$.

Theorem I. *A map $\mathfrak{F} \in \mathcal{G}_1$ has a uniformly elliptic, parabolic, or hyperbolic point if and only if every operator M in $\mathbf{B}^0$ is a uniformly elliptic, parabolic, or hyperbolic point for $\mathfrak{F}$.*

Proof. Elliptic case: Suppose that $M \in \mathbf{B}^0$ is not a uniformly elliptic point for $\mathfrak{F}$. Then there is a sequence of vectors $x_N \in H_+$ with $\|x_N\| = 1$ such that $\|\mathfrak{F}^N(M)x_N\| \to 1$. If $K \in \mathbf{B}^0$, then inequality (6) implies that $\|\mathfrak{F}^N(K)x_N - \mathfrak{F}^N(M)x_N\| \to 0$. Therefore $\|\mathfrak{F}^N(K)x_N\| \to 1$, and so K is not an elliptic point for $\mathfrak{F}$.

Parabolic Case: Suppose that M is a uniformly parabolic point for $\mathfrak{F}$. If $K \in \mathbf{B}^0$, then inequality (6) implies that $\|\mathfrak{F}^{\pm N}(K) - \mathfrak{F}^{\pm N}(M)\| \to 0$ and thus $\mathfrak{F}^{\pm N}(K)$ is invertible for large N and $\|[\mathfrak{F}^{\pm N}(K)]^{-1}\| \to 1$. Furthermore,

(7) $$\|\mathfrak{F}^N(K) - \mathfrak{F}^N(K)\| \leqq \|\mathfrak{F}^N(K) - \mathfrak{F}^N(M)\| + \|\mathfrak{F}^N(M) - \mathfrak{F}^{-N}(M)\| + \|\mathfrak{F}^{-N}(M) - \mathfrak{F}^{-N}(K)\|.$$

Inequality (6) and the fact that M is a uniformly parabolic point for $\mathfrak{F}$ imply that the right hand side of inequality (7) converges to 0. Therefore K is a uniformly parabolic point of F.

The Hyperbolic Case is proved similarly.

Definition. A map $\mathfrak{F} \in \mathcal{G}$ is called uniformly elliptic, parabolic or hyperbolic if and only if it has a uniformly elliptic, parabolic or hyperbolic point, respectively.

Fixed point theorems

Theorem II. *A map $\mathfrak{F}$ in $\mathcal{G}_1$ is uniformly elliptic if and only if $\mathfrak{F}$ has a fixed point in the interior of* **B**.

Proof. The following is a consequence of Theorem 6. 1 [5] due to R. S. PHILLIPS:

(8) If U is a Q-unitary operator, then $\|U^{\pm N}\| < M$ for all N if and only if U has an invariant maximal positive subspace P with the property $P + P' = H$.

We now prove the equivalence of the Theorem II and (8). Suppose that U corresponds to $\mathfrak{F}$ as in (4) with the matrix representation for U given by (2). Since U is Q-unitary, $U^{-1} = [E_+ - E_-]U^*[E_+ - E_-]$ and an easy computation shows that for $x \in H_+$ and $y \in H_-$ we have

$$\|U^{-1}[x+y]\|^2 = \|\psi^*(1-J^*J)^{-\frac{1}{2}}x - \psi^*(1-J^*J)^{-\frac{1}{2}}J^*y\|^2 +$$
$$+ \|-\varphi^*(1-JJ^*)^{-\frac{1}{2}}Jx + \varphi^*(1-JJ^*)^{-\frac{1}{2}}y\|^2 =$$
$$= \|(1-J^*J)^{-\frac{1}{2}}[x - J^*y]\|^2 + \|(1-JJ^*)^{-\frac{1}{2}}[y - Jx]\|^2$$

where $J = \mathfrak{F}(0)$. Consequently

$$\frac{1}{1-\|J_N\|^2} \leqq \|U^{-N}\|^2 \leqq 8\frac{1}{1-\|J_N\|^2}.$$

Thus $U^{\pm N}$ is uniformly bounded if and only if $\|\mathfrak{F}^{\pm N}(0)\| \leqq \alpha < 1$ and hence if and only if $\mathfrak{F}$ is uniformly elliptic. Now Lemma 6. 3 [5] says that a maximal positive subspace P has the property $P + P' = H$ if and only if $P \sim J$ and $\|J\| < 1$. These last two facts when combined with the fact that the Q-unitary operator U corresponding to $\mathfrak{F}$ has an invariant maximal positive subspace P if and only if the contraction J corresponding to P is fixed by $\mathfrak{F}$ imply that Theorem II and statement (8) are equivalent.

It is not known if hyperbolic and parabolic maps have fixed points. We shall now consider commuting families of general symplectic maps. Suppose $\mathcal{S}$ is a subgroup of $\mathcal{G}_1$ and $\Gamma_{\mathcal{S}} = \{U\colon U \text{ corresponds to } \mathfrak{F} \text{ and } \mathfrak{F} \in \mathcal{S}\}$. The group $\mathcal{S}$ is commutative if and only if the group $\Gamma_{\mathscr{S}}$ is *scalar commutative* (cf. sec. Ia. [2]) i.e. if $U, V \in \Gamma_{\mathscr{S}}$ then there is a number β with $|\beta|=1$ such that $UV=\beta VU$. A scalar commutative group $\mathscr{S}$ of operators is called *full* if $\alpha U \in \mathscr{S}$ whenever $U \in \mathscr{S}$ and α is a scalar with $|\alpha|=1$. The *group* $\mathscr{S}$ will be called *elliptic* if for each $x \in H$ there is a number $a(x)<1$ such that $\|\mathfrak{F}(0)x\| \leqq a(x)\|x\|$ for all $\mathfrak{F} \in \mathscr{S}$; the group $\mathscr{S}$ will be called *uniformly elliptic* if $a(x)<a<1$ for all $x \in H$.

Theorem III. *A commuting group $\mathscr{S}$ of general symplectic maps is uniformly elliptic if and only if $\mathscr{S}$ has a fixed point in the interior of* $\mathbf{B}$.

Proof. We must prove statement (8) not for a single Q-unitary map U but for a scalar commuting family $\Gamma_{\mathscr{S}}$ of Q-unitary operators. It is clear from the original proof of (8) in [5] that any group Γ of Q-unitary operators has an invariant positive subspace P with $P+P' = H$ if and only if there is a bounded invertible operator B such that BUB^{-1} is unitary for each $U \in \Gamma$. The proof of Theorem II implies that $\mathscr{S}$ is uniformly elliptic if and only if $\Gamma_{\mathscr{S}}$ is uniformly bounded. Thus we need only prove

Lemma. *If Γ is a full, scalar commuting group of operators which is uniformly bounded, then Γ is similar to a group of unitary operators.*

Proof. The proof in the case where Γ is commutative involves finding an invariant mean on Γ. The case at hand requires just a slight modification of this. Although Γ is not commutative, $\Gamma/T = \Gamma$ modulo the circle group T is commutative. Thus there is an invariant mean on Γ/T (for instance see [1]). For fixed $x, y \in H$ the function f on Γ/T defined by $f(\tilde{U})=(Ux, Uy)$ where $\tilde{U} \in \Gamma/T$ and $U \in \Gamma$ is any element in the equivalence class $\tilde{U}$ is bounded. Thus we may define a bilinear form $(\ ,\ )'$ on it by

$$(x, y)'=m(f).$$

Since m is an invariant mean each $U \in \Gamma$ is unitary with respect to $(\ ,\ )'$ and it is easy to see that $\|x\|'=\sqrt{(x,x)'}$ is equivalent to the original norm on H. The lemma is immediate from this.

Inequality (6) yields the following lemma for the non-elliptic case.

Theorem IV. *If $\mathscr{S}$ is a commutative group of maps in $\mathscr{G}_1$ which is not elliptic and if for each $\mathfrak{F} \in \mathscr{S}$ the operator $\mathfrak{F}(0)$ is compact, then $\Gamma_{\mathscr{S}}$ has a non-trivial positive invariant subspace.*

Proof. The condition $\mathfrak{F}(0)$ is compact is equivalent to $\mathfrak{F}$ being continuous in the weak operator topology (cf. [3] and the author's Stanford dissertation).

Since $\mathscr{S}$ is not elliptic there is a sequence $\mathfrak{F}_N \in \mathscr{S}$ and a vector x such that $\|\mathfrak{F}_N(0)x\| \to 1$. Since $\mathbf{B}$ is compact in the weak operator topology we may assume that $\mathfrak{F}_N(0) \to T$ in the weak operator topology. If $\mathfrak{G} \in \mathscr{S}$, then $\mathfrak{G}[\mathfrak{F}_N(0)] \to \mathfrak{G}(T)$

in the weak operator topology; however by inequality (6)

$$\|\mathfrak{G}(\mathfrak{F}_N(0))x-\mathfrak{F}_N(0)x\| = \|\mathfrak{F}_N(\mathfrak{G}(0))x-\mathfrak{F}_N(0)x\| \to 0.$$

Thus $\mathfrak{G}(T)x=Tx$. Let $p = x+Tx$, let U be a Q-unitary operator which corresponds to $\mathfrak{G}$, and let $P\sim T$. Then $p = x+Tx\in P$ and property (4) implies that $p = x+\mathfrak{G}(T)x\in UP$. Therefore $p\in S = \bigcap_{U\in\Gamma_{\mathcal{S}}} UP$ and S is non-trivial. The form of S implies that it is invariant under operators in $\Gamma_{\mathcal{S}}$ and that S is positive.

Spaces with H_+ finite dimensional (Pontryagin spaces)

We give a new proof of:

Theorem. *If H_+ is finite dimensional and if $\mathcal{S}$ is a commutative subgroup of $\mathcal{G}_1$, then $\mathcal{S}$ has a fixed point.*

Proof. Let P be a subspace of H which is maximal with respect to being positive and invariant under $\Gamma_{\mathcal{S}}$. By Naimark's arguments in [4] it suffices to prove that $\Gamma_{\mathcal{S}}$ restricted to P' or to an appropriate modification of P' has a non-trivial invariant positive subspace. In effect it suffices for us to prove that $\Gamma_{\mathcal{S}}$ has a non-trivial invariant positive subspace.

Since H_+ is finite dimensional either $\Gamma_{\mathcal{S}}$ is uniformly elliptic or $\Gamma_{\mathcal{S}}$ is not elliptic. Theorem III and Theorem IV imply that $\Gamma_{\mathcal{S}}$ has a non-trivial invariant positive subspace in either case.

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