A remark on the cosine of linear operators

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1. In their recent note [2], K. GUSTAFSON and B. ZWAHLEN proved that an unbounded linear operator T acting in a pre-Hilbert space has cosine zero. It is our purpose to show that this statement can be extended to the case of unbounded linear mappings T from a complex (real) normed vector space X into a normed vector space Y, provided there is given a sesquilinear form $Q: X \times Y \to \mathbf{C}(\mathbf{R})$ such that

$$|Q(x, y)| \leqq \|x\| \, \|y\| \tag{1}$$

for all $x \in X$, $y \in Y$. The *cosine* of a mapping T from X to Y with respect to Q is then defined by

$$\cos_Q(T) = \inf \frac{|Q(x, Tx)|}{\|x\| \, \|Tx\|},$$

where the infimum is taken over all x in the domain $D(T)$, with $x \neq 0$, $Tx \neq 0$.

Theorem. *If to the linear operator $T: D(T) \subset X \to Y$ there exists a sesquilinear form Q such that $\cos_Q(T) > 0$, then T is bounded.*

The proof of the theorem is devided into two parts. We first introduce the concept of quasi-boundedness, which is due to F. E. BROWDER and the writer, and which turned out to be extremely useful in the study of nonlinear mappings of monotone type [1]. The mapping T is said to be *quasi-bounded with respect to the form Q*, if from the boundedness of the sequence $\{x_n\} \subset D(T)$ together with the boundedness of the sequence $\{Q(x_n, Tx_n)\}$ it follows that $\{Tx_n\}$ remains bounded. We prove that for an operator T which is homogeneous of some positive degree k (i.e. $D(T)$ a cone and $T(\mu x) = \mu^k T(x)$ for $\mu > 0$, $x \in D(T)$), quasi-boundedness implies boundedness. This observation allows us to give a proof of the theorem which seems to be more transparent even in the particular situation discussed in [2].

A closing example shows that the existence of a form Q with $\cos_Q(T) > 0$ is *not* necessary for the boundedness of a linear mapping T.

2. We shall preface the proof of the theorem with the following

Lemma. *Let the mapping $T: D(T)\subset X\to Y$ be homogeneous of degree $k>0$, and suppose there exists a sesquilinear form Q such that T is quasi-bounded with respect to Q. Then T maps bounded sets in X onto bounded sets in Y.*

Proof. For $\lambda>0$, let

$$f(\lambda)=\sup\{\|Tu\|:\ u\in D(T),\ \|u\|\leqq 1,\ |Q(u, Tu)|\leqq\lambda\}.$$

Because of the quasi-boundedness of T, f is a well-defined increasing function. We observe that for $\lambda\geqq 1$,

$$f(\lambda)\leqq\lambda^{\frac{k}{1+k}}f(1).$$

Hence

$$f(\lambda)\leqq\lambda^{\frac{k}{1+k}}f(1)+f(1),\qquad \lambda>0.$$

For $x\in D(T)$ with $\|x\|\leqq 1$ we set $\lambda=|Q(x, Tx)|$ and get

$$\|Tx\|\leqq f(|Q(x, Tx)|)\leqq\|Tx\|^{\frac{k}{1+k}}f(1)+f(1).$$

This estimate implies the boundedness of T, q.e.d.

Proof of the Theorem. In virtue of the lemma, it suffices to prove that T is quasi-bounded with respect to Q.

Assume that $\{x_n\}\subset D(T)$ is a sequence with $\|x_n\|\leqq c$, $|Q(x_n, Tx_n)|\leqq c$, but $\|Tx_n\|\to\infty$. Since $\|x_n\|\,\|Tx_n\|\cos_Q(T)\leqq|Q(x_n, Tx_n)|\leqq c$, we infer that[1] $x_n\to 0$. We construct a bounded sequence $\{u_n\}\subset D(T)$ such that $Q(u_n, Tx_n)=0$ and $\{Tu_n\}$ is bounded. For this purpose, let a and b be linearly independent vectors of $D(T)$ with $\|a\|=\|b\|=1$,[2] and for each n set $u_n=\alpha_n a+\beta_n b$, where α_n and β_n are solutions of the equations $|\alpha_n|^2+|\beta_n|^2=1$, $\alpha_n Q(a, Tx_n)+\beta_n Q(b, Tx_n)=0$. The function g: $[\alpha,\beta]\to g(\alpha,\beta)=\|\alpha a+\beta b\|$ is continuous, hence it admits its supremum and infimum on the (compact) unit sphere $|\alpha|^2+|\beta|^2=1$. Because of the linear independence of a and b, the infimum is positive. Consequently there exists $\gamma>0$ such that $\gamma^{-1}\leqq\|u_n\|\leqq\gamma$ for all n. In addition, $\|Tu_n\|\leqq\|Ta\|+\|Tb\|$. Setting $w_n=$ $=x_n+u_n\in D(T)$, we obtain

$$\frac{|Q(w_n, Tw_n)|}{\|w_n\|\,\|Tw_n\|}\leqq\frac{|Q(x_n, Tx_n)|+|Q(u_n, Tx_n)|+|Q(x_n, Tu_n)|+|Q(u_n, Tu_n)|}{|\|u_n\|-\|x_n\||\cdot|\|Tx_n\|-\|Tu_n\||},$$

where the right hand side converges to 0 as $n\to\infty$, since the numerator remains

[1]) We use the symbols "$\to$" and "$\rightharpoonup$" to denote *strong* and *weak* convergence, respectively.

[2]) If $D(T)$ is one dimensional, the theorem is trivial.

bounded and the denominator tends to $+\infty$. We are thus led to a contradiction to the assumption $\cos_Q(T)>0$, q.e.d.

That $\cos_Q(T)>0$ for some form Q is not necessary for the boundedness of the linear mapping T, is shown by the following

Example. *Let T be a bounded linear operator from X to Y, and suppose there exists a sequence $\{x_n\}\subset D(T)$ with $\|x_n\|=1$, $x_n\rightharpoonup 0$, $Tx_n\neq 0$, such that the linear span of $\{Tx_n\}$ has finite dimension. Then $\cos_Q(T)=0$ for each form Q.*

References

[1] F. E. Browder and P. Hess, Nonlinear operators of monotone type in Banach spaces, *J. Functional Analysis* (to appear).

[2] K. Gustafson and B. Zwahlen, On the cosine of unbounded operators, *Acta Sci. Math.*, **30** (1969), 33—34.

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