Compact restrictions of operators

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1. Introduction. The purpose of this note is to set forth a definitive version of a theorem concerning operators on Hilbert space, and to discuss some consequences of that theorem that seem not to have been noticed before now. The theorem asserts that, unless an operator is, in a sense, nearly invertible, then it is "very small" on an infinite dimensional subspace. This fact has already been noted several times in the literature in one form or another (see, for example, [15, § 1. 2]; the main special case is valid even on Banach spaces [9, III. 1. 9]; for a version of the theorem valid in an infinite factor see [6], and the only thing in § 2 that can claim to be new is the manner in which we construe the notion of "very small". The results recounted in §§ 3—5 have greater claim to novelty.

Throughout this paper all *Hilbert spaces* will be complex, separable, and, unless the contrary possibility is explicitly stated, infinite dimensional. Furthermore, *operators* are always bounded, linear transformations from one Hilbert space into another. If $\mathscr{H}$ is a Hilbert space, then the algebra of all operators T from $\mathscr{H}$ into $\mathscr{H}$ will be denoted by $\mathscr{L}(\mathscr{H})$. We shall have occasion to refer to various ideals of operators, and we take this opportunity to remind the reader of the basic facts concerning the ideal structure of $\mathscr{L}(\mathscr{H})$. (By *ideal* we shall always mean two-sided ideal. Recall that $\mathscr{H}$ is assumed to be infinite dimensional; otherwise $\mathscr{L}(\mathscr{H})$ is simple.)

In the first place, every ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ satisfies the condition

$$\mathfrak{F}\subset\mathfrak{J}\subset\mathfrak{C},$$

where $\mathfrak{F}$ denotes the ideal of operators of finite rank and $\mathfrak{C}$ the ideal of all compact operators. From this it is immediately apparent that $\mathfrak{C}$ is the *only* proper norm-closed ideal in $\mathscr{L}(\mathscr{H})$. Non-closed ideals exist in great abundance, however, and have been completely described. Indeed, if C denotes the collection of all sequences $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers that tend to zero, then there is a simple one-to-one, inclusion preserving correspondence between the ideals $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ and the subsets J of C, called *ideal sets*, that satisfy the following conditions:

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i) if $\{\lambda_n\}$ is a sequence in J, and if π is any permutation of the positive integers, then $\{\lambda_{\pi(n)}\}$ is also in J,
ii) if $\{\lambda_n\}$ and $\{\mu_n\}$ are in J, then so is $\{\lambda_n+\mu_n\}$,
iii) if $\{\lambda_n\}$ is in J, and if $0\leqq\mu_n\leqq\lambda_n$ for all n, then $\{\mu_n\}$ is also in J.

The precise nature of this correspondence is as follows: if T belongs to $\mathfrak{J}$ then $|T|=(T^*T)^{\frac{1}{2}}$ does too, and, since $|T|$ is compact, its eigenvalues (counting multiplicities) can be arranged in a sequence belonging to C. The set of all sequences $\{\lambda_n\}$ so obtained from the various operators $T\in\mathfrak{J}$ forms the *ideal set* J of $\mathfrak{J}$. Conversely, if J is an ideal set in C, and if we say of an operator T on $\mathscr{H}$ that it *belongs to* J if, when the eigenvalues of $|T|$ are arranged in a sequence, that sequence belongs to J, then the set of all operators belonging to J forms an ideal $\mathfrak{J}$, of which J is clearly the ideal set. (These results are due originally to VON NEUMANN; a good account of them may be found in [5] or [7].) Note that under this correspondence the entire set C is the ideal set of the maximum ideal $\mathfrak{C}$ of all compact operators, and that the ideal set of the ideal $\mathfrak{F}$ of operators of finite rank is the set F of finitely non-zero sequences. Note also that these facts free the discussion of ideals in $\mathscr{L}(\mathscr{H})$ from the Hilbert space $\mathscr{H}$. When, in the sequel, we refer to an ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ and then to the "same" ideal on another space $\mathscr{K}$, what is meant, of course, is that ideal in $\mathscr{L}(\mathscr{K})$ having the same ideal set as $\mathfrak{J}$. Moreover, the correspondence between ideal sets and operators can be extended even to operators from one space to another. Let J be an ideal set of sequences and let $\mathfrak{J}$ be its associated ideal, and suppose given an operator T mapping one Hilbert space $\mathscr{H}$ into another space $\mathscr{K}$. Then we shall say that T is *affiliated with* $\mathfrak{J}$ if, when the eigenvalues of $|T|=(T^*T)^{\frac{1}{2}}$ are arranged in a sequence, that sequence belongs to J. (When $\mathscr{H}$ and $\mathscr{K}$ do coincide, affiliation reduces to set membership.) Note that if $T\colon\mathscr{H}\to\mathscr{K}$ is affiliated with $\mathfrak{J}$ in this sense, then it continues to be true that $T^*\ \mathscr{K}\to\mathscr{H}$ is also. Similarly, it is easy to show that if T_1 and T_2 both map $\mathscr{H}$ into $\mathscr{K}$ and if both are affiliated with $\mathfrak{J}$, then T_1+T_2 is too, and that if $T\colon\mathscr{H}\to\mathscr{K}$ is affiliated with $\mathfrak{J}$ and if $S_1\colon\mathscr{K}\to\mathscr{K}_1$, $S_2\colon\mathscr{H}_1\to\mathscr{H}$ so that the product S_1TS_2 is defined, then S_1TS_2 is also affiliated with $\mathfrak{J}$.

2. Operators with small restrictions. The following theorem is the central tool of the paper.

Theorem 2. 1. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, and let T be an operator mapping $\mathscr{H}$ into $\mathscr{K}$. Suppose that there does not exist a finite dimensional subspace $\mathscr{D}\subset\mathscr{H}$ such that $T|\mathscr{D}^\perp$ is bounded below. Then for any prescribed ideal $\mathfrak{J}$ other than the ideal $\mathfrak{F}$ of operators of finite rank, and for any η greater than zero, there exists an infinite dimensional subspace $\mathscr{L}\subset\mathscr{H}$ such that the restriction $T_0=T|\mathscr{L}\,(T_0\colon\mathscr{L}\to\mathscr{K})$ is affiliated with $\mathfrak{J}$ and satisfies the condition $\|T_0\|<\eta$.*

Before proving the theorem, it is advantageous to establish a working criterion for determining when an operator is affiliated with a given ideal.

Lemma 2. 2. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Then a necessary and sufficient condition for an operator $T: \mathscr{H} \to \mathscr{K}$ to be affiliated with a given ideal $\mathfrak{J}$ is that there exist an orthonormal basis $\{e_n\}$ in $\mathscr{H}$, and an orthonormal sequence $\{f_n\}$ in $\mathscr{K}$ such that $Te_n = \lambda_n f_n$ for all n, where $\{|\lambda_n|\}$ belongs to the ideal set of $\mathfrak{J}$.*

Proof. If the criterion is satisfied, then $|T|e_n = |\lambda_n|e_n$ for all n, so the condition is clearly sufficient. On the other hand, if T is affiliated with $\mathfrak{J}$, then there exists an orthonormal basis $\{e_n\}$ in $\mathscr{H}$ such that $|T|e_n = \lambda_n e_n$ for all n, where $\{\lambda_n\}$ is in the ideal set of $\mathfrak{J}$. But then, if W denotes the partial isometry in the polar resolution of T, so that $T = W|T|$, and if we set $f_n = We_n$, then $\{f_n\}$ is an orthonormal sequence in $\mathscr{K}$, and $Te_n = \lambda_n f_n$. □

Proof of Theorem 2. 1. If T has an infinite dimensional null space, we may simply set $T_0 = 0$. Otherwise, let $T = W|T|$ be the polar resolution of T as above, and let E denote the spectral measure of $|T|$. Then, according to our assumptions, no projection $E([0, \varepsilon))$ $(\varepsilon > 0)$ has finite rank, while $E(\{0\})$ does have finite rank. Hence $E((0, \varepsilon))$ has infinite rank for every positive ε, and it follows at once that for every positive ε there exists δ, $0 < \delta < \varepsilon$, such that $E((\delta, \varepsilon))$ has rank greater than one.

Now let $\{\lambda_n\}$ be any one fixed sequence in the ideal set J of $\mathfrak{J}$ satisfying the conditions $0 < \lambda_{n+1} \leqq \lambda_n < \eta$ for every n. (Such sequences exist since $J \neq F$; see [4, Lemma 1. 1].) We set $\varepsilon_1 = \lambda_1$ and determine δ_1, $0 < \delta_1 < \varepsilon_1$ such that $E_1 = E((\delta_1, \varepsilon_1))$ has rank exceeding one. Next, define $\varepsilon_2 = \delta_1 \wedge \lambda_2$ and choose δ_2 so that $0 < \delta_2 < \varepsilon_2$ and so that $E_2 = E((\delta_2, \varepsilon_2))$ has rank exceeding one. Continuing in this fashion, we obtain an infinite sequence of spectral projections E_n such that, for every n, $\mathscr{M}_n = E_n(\mathscr{H})$ has dimension at least two and such that $\| \, |T| \, |\mathscr{M}_n\| \leqq \lambda_n < \eta$. In each subspace $\mathscr{M}_n$ we select a pair of orthogonal unit vectors e_n and f_n in such a way that the plane $[e_n, f_n]$ contains the vector $|T|e_n$, and write

$$|T|e_n = \alpha_n e_n + \beta_n f_n.$$

Then $0 < \alpha_n = (|T|e_n, e_n) \leqq \lambda_n$ and $|\beta_n| \leqq 2\lambda_n$ for all n.

Finally, let $\mathscr{L}$ denote the subspace spanned by the sequence $\{e_n\}$, and set $A = |T| \, |\mathscr{L} \colon \mathscr{L} \to \mathscr{H}$, so that $T_0 = T|\mathscr{L}$ is given by $T_0 = WA$. Since the vectors Te_n are all orthogonal and less than η in norm, it is obvious that $\|T_0\|$ is also less than η. On the other hand, if P denotes the (orthogonal) projection of $\mathscr{H}$ onto $\mathscr{L}$, then PA and $(1-P)A$, regarded as mappings from $\mathscr{L}$ to $\mathscr{K}$, both clearly satisfy the criterion of Lemma 2. 2. But then, of course, $A = PA + (1-P)A$ and $T_0 = WA$ are also affiliated with $\mathfrak{J}$. □

The hypotheses of Theorem 2. 1 are formulated as they are in order to facilitate the proof of the theorem, not with a view to applications. We pause to list some alternate versions of the condition imposed on T.

Lemma 2. 3. *The following conditions are equivalent for any operator* $T: \mathscr{H} \to \mathscr{K}$.[2])

i) *T is bounded below on the orthocomplement of some finite dimensional subspace.*
ii) *The null space of T is finite dimensional and the range of T is closed.*
iii) *There exists an operator $S: \mathscr{K} \to \mathscr{H}$ such that ST is a projection of finite co-rank.*
iv) *T is semi-Fredholm with index less than $+\infty$.*
v) *There exists no orthonormal sequence $\{e_n\}_{n=1}^{\infty}$ such that $\|Te_n\| \to 0$.*

In the special case $\mathscr{H} = \mathscr{K}$ the conclusion of the main theorem can also be reformulated in a useful manner. The following is an immediate consequence of Theorem 2. 1, from which, in turn, the latter may easily be deduced.

Corollary 2. 4. *Let T be an operator in $\mathscr{L}(\mathscr{H})$ and suppose that the range of T is not closed, or that the null space of T is infinite dimensional. Let $\mathfrak{J}$ be any ideal other than the ideal $\mathfrak{F}$, and let η be a positive number. Then there exists a decomposition $\mathscr{H} = \mathscr{L} \oplus \mathscr{L}^{\perp}$ of $\mathscr{H}$ into infinite dimensional subspaces with respect to which the matrix representation of T has the form*

$$\begin{pmatrix} K & * \\ L & * \end{pmatrix}$$

where K and L are both affiliated with $\mathfrak{J}$ and have norm less than η.

Proof. From the proof of Theorem 2. 1 it is clear that both the subspace $\mathscr{L}$ constructed there and its orthocomplement are infinite dimensional. Everything else is obvious. □

3. Subspaces that are nearly invariant. If $\mathfrak{J}$ is any ideal in $\mathscr{L}(\mathscr{H})$, then the quotient algebra $\mathscr{L}(\mathscr{H})/\mathfrak{J}$ is clearly a *-algebra. Moreover, for the norm-closed ideal $\mathfrak{C}$ of all compact operators the quotient algebra is even a C^*-algebra with respect to the quotient norm. As is customary, we shall refer to this algebra as the *Calkin algebra* over $\mathscr{H}$. If T is an operator in $\mathscr{L}(\mathscr{H})$, we denote by $\hat{T}$ the residue class of T in the Calkin algebra.

[2]) This lemma is but a part of a more encompassing theorem due to J. P. Williams [14, Theorem (1. 1)], which generalizes some results of Wolf [15]. The authors wish to take this opportunity to express this gratitude to Williams for a number of stimulating and enlightening conversations on this point as well as on other related subjects.

Theorem 3. 1. *Let T be an operator $\mathscr{L}(\mathscr{H})$, and let $\mathfrak{I}$ be any ideal other than $\mathfrak{F}$. Then there exists a scalar λ and a decomposition of $\mathscr{H}$ into infinite dimensional subspaces $\mathscr{L}$ and $\mathscr{L}^{\perp}$ such that the corresponding matrix representation of T has the form*

$$\begin{pmatrix} \lambda+K & * \\ L & * \end{pmatrix} \tag{1}$$

where K and L are both affiliated with $\mathfrak{I}$. Moreover, the decomposition can be so arranged that the norms of K and L are less than any prescribed positive η.

Proof. The residue class $\hat{T}$ of T in the Calkin algebra over $\mathscr{H}$ has non-empty spectrum σ by the Gelfand—Mazur Theorem, and in σ there are points λ such that $\hat{T}-\lambda$ has no left inverse. (These are the points of the *left essential spectrum* in the terminology of [14]. For example, any complex number in the topological boundary of σ is such a λ.) But then $T-\lambda$ fails to satisfy the criterion of Lemma 2. 3, and the theorem follows. □

As the proof of Theorem 3. 1 shows, the choice of λ is quite independent of $\mathfrak{I}$ and of η. It may be noted that λ can be taken to be any scalar in the boundary of the spectrum of T itself, other than an isolated eigenvalue of finite multiplicity, since such points automatically survive in the spectrum of $\hat{T}$; see, for instance, [10, Theorem 2]. It may also be noted that Theorem 3. 1, as well as Corollaries 3. 2, 3. 5, and 3. 6, are definitely false for $\mathfrak{I}=\mathfrak{F}$. Finally, if $\mathscr{L}$ and $\mathscr{L}^{\perp}$ are both identified with the same space $\mathscr{K}$ (as they may be whenever convenience so dictates), then the entries in (1) will all be in $\mathscr{L}(\mathscr{K})$, and K and L will be actual members of the ideal $\mathfrak{I}$ on $\mathscr{K}$.

Theorem 3. 1 may be paraphrased by saying that the residue class of T modulo $\mathfrak{I}$ has the form

$$\begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}.$$

In this formulation, however, the matrix entries are to be interpreted merely as the components in the Pierce decomposition of the residue class of T relative to a non-zero, Hermitian idempotent; residue classes modulo $\mathfrak{I}$ cannot, in general, be realized spatially as operators.

Corollary 3. 2. *For any operator T in $\mathscr{L}(\mathscr{H})$, and for any ideal $\mathfrak{I}$ in $\mathscr{L}(\mathscr{H})$ other than $\mathfrak{F}$, there exists an infinite dimensional subspace $\mathscr{L}$ with infinite dimensional orthocomplement $\mathscr{L}^{\perp}$ such that $\mathscr{L}$ is invariant under T modulo $\mathfrak{I}$, i.e., such that $(1-P)TP\in\mathfrak{I}$, where P denotes the projection of $\mathscr{H}$ onto $\mathscr{L}$.*

Note, in particular, that Corollary 3. 2 solves in the affirmative the invariant subspace problem in the Calkin algebra. (For another representation of $\mathscr{L}(\mathscr{H})$

having the same property the reader may consult [1].) The following result exploits the metrical aspect of Theorem 3. 1.

Corollary 3. 3. *For any operator T in $\mathcal{L}(\mathcal{H})$ and any positive number η there exists an operator R such that $\|T-R\| < \eta$ and such that R possesses an infinite dimensional invariant subspace $\mathcal{L}$ having infinite dimensional orthocomplement. Likewise, for any positive integer p, there exists an operator R_p that is within η of T in norm and possesses a p-dimensional invariant subspace.*

Proof. By Theorem 3. 1 there exists an infinite dimensional subspace $\mathcal{L}$ with infinite dimensional orthocomplement such that the corresponding matrix representation has the form (1) with the property that $\|L\| < \eta$. To obtain a suitable operator R we have but to define

$$R = T - \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}.$$

In order to construct R_p we choose bases $\{e_n\}$ and $\{f_n\}$ in $\mathcal{L}$ and $\mathcal{L}^{\perp}$, respectively. It is then a simple matter, since K and L are compact, to find p basis vectors e_n such that, if $\mathcal{P}$ denotes the subspace they span, then $\|(T-\lambda)|\mathcal{P}\| < \eta$. Then the matrix of R_p may be obtained by replacing all the off-diagonal entries in the correspondings columns by zero's. □

In the special case of a seminormal operator the preceding results can be improved in a natural but significant manner. First, a lemma.

Lemma 3. 4. *Let S and T be two operators from $\mathcal{H}$ into $\mathcal{K}$, and suppose that S is metrically dominated by T, i.e., that $\|Sx\| \leqq \|Tx\|$ for every x in $\mathcal{H}$. Then S is affiliated with every ideal with which T is.*

Proof. It is clear that $|S|$ is metrically dominated by $|T|$. The lemma follows via a straightforward application of the minimax principle, or alternatively, via [8, Theorem 1]. □

Theorem 3. 5. *Let T be a seminormal operator in $\mathcal{L}(\mathcal{H})$, and let $\mathfrak{J}$ be any ideal other than $\mathfrak{F}$. Then there exists a scalar λ and a decomposition of $\mathcal{H}$ into infinite dimensional subspaces $\mathcal{L}$ and $\mathcal{L}^{\perp}$ such that the corresponding matrix representation of T has the form*

$$\begin{pmatrix} \lambda + K & M \\ L & * \end{pmatrix} \tag{2}$$

where K, L, and M all are affiliated with $\mathfrak{J}$. Moreover, the decomposition can be so arranged that the norms of K, L, and M are all less than any prescribed positive η.

Proof. We may suppose that T is hyponormal. Let $\mathscr{H}$ be decomposed as in Theorem 3. 1, in such a way that, in the matrix representation (1), the operator

$$\begin{pmatrix} K & 0 \\ L & 0 \end{pmatrix}$$

has norm less than η. Since K and L are affiliated with $\mathfrak{I}$, it follows, as we have seen, that $(T-\lambda)|\mathscr{L}: \mathscr{L} \to \mathscr{H}$ is affiliated with $\mathfrak{I}$ and has norm less than η. Since $T-\lambda$ is hyponormal along with T, this implies in turn, by Lemma 3. 4, that $(T-\lambda)^*|\mathscr{L}$ is also affiliated with $\mathfrak{I}$ and has norm less than η. Since the matrix of $(T-\lambda)^*$ is

$$\begin{pmatrix} K^* & L^* \\ M^* & * \end{pmatrix}$$

it follows, finally, that M and M^* are affiliated with $\mathfrak{I}$ and have norm less than η. □

Here again, as was the case in Theorem 3. 1, the result may be interpreted matricially if we are careful not to attribute undue spatial significance to the matrix entries. It says that if $\mathfrak{I} \neq \mathfrak{F}$, and if T is seminormal, then the residue class of T modulo $\mathfrak{I}$ has the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}. \tag{3}$$

(In this connection see also [14, Theorem (4. 2)].)

Corollary 3. 6. *If T is a seminormal operator in $\mathscr{L}(\mathscr{H})$, and if $\mathfrak{I}$ is any ideal in $\mathscr{L}(\mathscr{H})$ other than $\mathfrak{F}$, then there exists an infinite dimensional subspace $\mathscr{L}$, with infinite dimensional orthocomplement, such that $\mathscr{L}$ is reducing for T modulo $\mathfrak{I}$, i.e., such that $TP-PT \in \mathfrak{I}$ where P denotes the projection of $\mathscr{H}$ onto $\mathscr{L}$.*

Corollary 3. 7. *For any seminormal operator T in $\mathscr{L}(\mathscr{H})$ and any positive number η there exists an operator R such that $\|T-R\| < \eta$ and such that R possesses an infinite dimensional reducing subspace with infinite dimensional orthocomplement. Likewise, for any positive integer p, there exists an operator R_p that is within η of T in norm and possesses a p-dimensional reducing subspace.*

The proofs of Corollaries 3. 6 and 3. 7 are straightforward analogs of those of Corollaries 3. 2 and 3. 3, and will be omitted. The finite dimensional part of Corollary 3. 7 is essentially due to Stampfli [12], who states the result in the case $p=1$. We owe to the same paper the observation that Corollary 3. 7 remains valid if T merely differs from a seminormal operator by a compact operator. (The same may also be said, of course, of Corollary 3. 3.)

Theorem 3. 5 yields at least one other interesting result. Indeed, a glance at (3) reveals the validity of the following assertion.

Corollary 3. 8. *If T is a seminormal operator in $\mathcal{L}(\mathcal{H})$, and if $\mathfrak{I}$ is any ideal other than $\mathfrak{F}$, then there exists an infinite dimensional subspace $\mathcal{L}$ such that, for every X in $\mathcal{L}(\mathcal{H})$, the commutator $C = TX - XT$ has the property that its compression $PC|\mathcal{L}$ to $\mathcal{L}$ belongs to $\mathfrak{I}$.*

In particular, this shows that 0 belongs to the (essential) numerical range of C (see [13]), thus recapturing a result of C. R. PUTNAM [11].

4. Operators congruent to scalars. In this section we give several criteria for an operator in $\mathcal{L}(\mathcal{H})$ to be congruent to a complex number modulo one or another of the ideals in $\mathcal{L}(\mathcal{H})$.

Theorem 4. 1. *Let T be an operator in $\mathcal{L}(\mathcal{H})$ and let $\mathfrak{I}$ be an ideal. Then a necessary and sufficient condition for T to be congruent to a scalar modulo $\mathfrak{I}$ is that, for any two orthogonal subspaces $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{H}$,*

(C) $P_{\mathcal{N}}TP_{\mathcal{M}} \in \mathfrak{I}$, *where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the (orthogonal) projections of $\mathcal{H}$ onto $\mathcal{M}$ and $\mathcal{N}$, respectively.*

Proof. The necessity of the condition is evident. To prove sufficiency, consider first the case $\mathfrak{I} \neq \mathfrak{F}$. According to Theorem 3. 1, there exist subspaces $\mathcal{L}$ and $\mathcal{L}^{\perp}$, both infinite dimensional, with respect to which T has the form

$$\begin{pmatrix} \lambda + K & X \\ L & Y \end{pmatrix}$$

with K and L affiliated with $\mathfrak{I}$. Moreover, X is also affiliated with $\mathfrak{I}$ because of (C). Hence, T is congruent modulo $\mathfrak{I}$ to the matrix

$$T' = \begin{pmatrix} \lambda & 0 \\ 0 & Y \end{pmatrix}.$$

Now let V be an isometry of $\mathcal{L}^{\perp}$ onto $\mathcal{L}$, and use the map $1 \oplus V$ to identify $\mathcal{H}$ with $\mathcal{L} \oplus \mathcal{L}$. Under this unitary equivalence, T' is carried onto the operator

$$T'' = \begin{pmatrix} \lambda & 0 \\ 0 & Y_0 \end{pmatrix}$$

where $Y_0 = VYV^*$. Clearly T'' continues to satisfy (C), so that if $\mathcal{M}$ and $\mathcal{N}$ denote, respectively, the subspaces $\{(x, x): x \in \mathcal{L}\}$ and $\{(x, -x): x \in \mathcal{L}\}$, then $P_{\mathcal{N}}T''P_{\mathcal{M}}$ must belong to $\mathfrak{I}$. But for any vector (x, y) in $\mathcal{L} \oplus \mathcal{L}$ we have $P_{\mathcal{N}}(x, y) = \frac{1}{2}(x - y, y - x)$, so that

$$P_{\mathcal{N}}T''(x, x) = \tfrac{1}{2}((\lambda - Y_0)x, (Y_0 - \lambda)x).$$

It follows at once that Y_0 is congruent to λ modulo $\mathfrak{I}$, and hence that T'' and T' are too.

It remains to consider the case $\mathfrak{J}=\mathfrak{F}$. If T satisfies (C) with $\mathfrak{J}=\mathfrak{F}$, then, by what has already been shown, T is congruent to some λ modulo every ideal $\mathfrak{J}\neq\mathfrak{F}$ (clearly the same λ in each case), so that $T-\lambda$ belongs to the intersection of all the ideals $\mathfrak{J}\neq\mathfrak{F}$. Since this intersection is known to be equal to $\mathfrak{F}$ (see [4]), the theorem follows. □

A second criterion is given by the following corollary.

Corollary 4. 2. *A necessary and sufficient condition for an operator T in $\mathcal{L}(\mathcal{H})$ to be congruent to some scalar modulo a given ideal $\mathfrak{J}$ is that for every infinite dimensional subspace $\mathcal{L}$ with infinite dimensional complement, the compression $P_{\mathcal{L}}T|\mathcal{L}$ of T to $\mathcal{L}$ should be congruent modulo $\mathfrak{J}$ to some scalar.*

Proof. Once again, it is clear that the condition is necessary. The proof will be completed by showing that an operator T satisfying the hypothesis of the corollary also satisfies condition (C) of Theorem 4. 1. Accordingly, let $\mathcal{M}$ and $\mathcal{N}$ be orthogonal subspaces of $\mathcal{H}$. Clearly we may assume both $\mathcal{M}$ and $\mathcal{N}$ to be infinite dimensional, since otherwise $P_{\mathcal{N}}TP_{\mathcal{M}}$ is automatically in $\mathfrak{F}$. Write $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are both infinite dimensional, and consider the compression of T to $\mathcal{M}_1\oplus\mathcal{N}$. The hypothesis assures us that this compression is congruent to some scalar modulo $\mathfrak{J}$, whence, by Theorem 4. 1, $P_{\mathcal{N}}TP_{\mathcal{M}_1}$ must belong to $\mathfrak{J}$. Similarly, $P_{\mathcal{N}}TP_{\mathcal{M}_2}$ belongs to $\mathfrak{J}$, from which it follows immediately that $P_{\mathcal{N}}TP_{\mathcal{M}}$ does so too. □

Our third and final criterion is one that has already essentially been noted by CALKIN (see [5, Theorem 2. 9]) but our proof is completely different from his.

Theorem 4. 3. *A necessary and sufficient condition for an operator T in $\mathcal{L}(\mathcal{H})$ to be congruent to a scalar modulo an ideal $\mathfrak{J}$ is that $TX-XT$ should belong to $\mathfrak{J}$ for every X in $\mathcal{L}(\mathcal{H})$.*

Proof. As before, the condition is clearly necessary, and we verify its sufficiency by showing that an operator that satisfies it also satisfies condition (C). Let $\mathcal{M}$ and $\mathcal{N}$ be orthogonal subspaces of $\mathcal{H}$ (infinite dimensional as before), and let W be any partial isometry with initial space $\mathcal{N}$ and final space $\mathcal{M}$. Then $(TW-WT)P_{\mathcal{M}}$ belongs to $\mathfrak{J}$ along with $TW-WT$, and since $W|\mathcal{M}=0$, this implies that $WTP_{\mathcal{M}}$ belongs to $\mathfrak{J}$. But then so does $P_{\mathcal{N}}WTP_{\mathcal{M}}=WP_{\mathcal{N}}TP_{\mathcal{M}}$ and therefore, finally, $W^*WP_{\mathcal{N}}TP_{\mathcal{M}}=P_{\mathcal{N}}TP_{\mathcal{M}}$. □

It may be noted that in the special case $\mathfrak{J}=\mathfrak{C}$ all three of these results yield criteria for an operator not to be a commutator [3]. This observation, Theorem 4. 3, and also the final result of § 3 all suggest that the ideas of the present note have interesting ramifications into commutator theory. In the next and final section we explore these connections in some depth.

5. Applications to commutator theory. As has just been noted, it is shown in [3] that an operator T in $\mathcal{L}(\mathcal{H})$ is a commutator if and only if it is not congruent to a non-zero scalar modulo the ideal $\mathfrak{C}$. On the other hand, in the earlier paper [2] it was shown, using considerably more elementary techniques, that every operator on $\mathcal{H}\oplus\mathcal{H}$ of the form

$$\begin{pmatrix} * & K_1 \\ * & K_2 \end{pmatrix}$$

where K_1 and K_2 are compact operators, is a commutator. Considering this fact, together with Theorem 3. 1, and taking adjoints if necessary, we immediately obtain the following result.

Theorem 5. 1. *Every non-Fredholm operator in $\mathcal{L}(\mathcal{H})$ is a commutator.*

This theorem prompts the following question: how far is it possible to proceed with the solution of the commutator problem, using only the techniques of [2] and the results of § 2? In other words, how far can one proceed without use of the sophisticated results of [3]; in particular, without introduction of the η-function and the standard form for operators of class (F)?

It is almost certain that one should not expect much success with the Fredholm operators of index zero, since the non-commutators in $\mathcal{L}(\mathcal{H})$ are Fredholm of index zero, while, at the same time, there are many Fredholm operators of index zero that are commutators, e.g., the invertible operators of class (F). Thus it is reasonable to limit attention to Fredholm operators of index different from zero. Operating under the above named restrictions, we are able to prove the following suggestive result.

Theorem 5. 2. *Every partial isometry in $\mathcal{L}(\mathcal{H})$ that is a Fredholm operator of index different from zero is a commutator.*

Proof. Note first that consideration of adjoints shows that it suffices to deal with the case in which the given partial isometry W has negative index. In this case there exists an operator F of finite rank (possibly zero) such that $V+F$ is an isometry, and such that the ranges of F and W are orthogonal. The isometry $W+F$ can be written uniquely as $W+F = U\oplus S$, where U is a unitary operator on a k-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ $(0\leqq k\leqq\aleph_0)$, while S is a unilateral shift of multiplicity m $(0<m<\aleph_0)$ acting on the space $\mathcal{M} = \mathcal{H}\ominus\mathcal{K}$. Suppose, temporarily, that $m=1$, and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathcal{M}$ such that $Se_n=e_{n+1}$ for all n. Reordering this basis as

$$\{e_1, e_3, \dots, e_{2n-1}, \dots;\ e_2, e_4, \dots, e_{2n}, \dots\}$$

we obtain a unitary isomorphism of $\mathcal{M}$ onto a Hilbert space $\mathcal{N}\oplus\mathcal{N}$, which carries S onto an operator matrix of the form

$$\begin{pmatrix} 0 & S_0 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

where S_0 is unitarily equivalent with S. A similar device shows that, no matter what the multiplicity m may be, S is always unitarily equivalent with (4), where S_0 is unitarily equivalent with S itself. It follows easily that $W+F = U\oplus S$ is unitarily equivalent with an operator matrix

$$\begin{pmatrix} U_1 & S_1 \\ B_1 & 0 \end{pmatrix} \tag{5}$$

acting on a Hilbert space $\mathcal{P}\oplus\mathcal{P}$, where U_1 is the direct sum of a unitary operator and the zero operator on an infinite dimensional space, while S_1 is an isometry and B_1 is a co-isometry. (If $k=0$, then $U_1=0$, if k is finite, then U_1 has finite rank, and, if $k=\aleph_0$, then S_1 has infinite defect.) Now the unitary isomorphism φ of $\mathcal{H}$ onto $\mathcal{P}\oplus\mathcal{P}$ that carries $W+F = U\oplus S$ onto (5) also carries F onto some matrix, — say the matrix

$$\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}.$$

Clearly each F_i $(i=1, 2, 3, 4)$ is of finite rank, and clearly also the given partial isometry W is unitarily equivalent via φ with an operator W_0 having the matrix

$$\begin{pmatrix} U_1-F_1 & S_1-F_2 \\ B_1-F_3 & -F_4 \end{pmatrix}. \tag{6}$$

Since the range of W is orthogonal in $\mathcal{H}$ to the range of F, it follows easily that the null space in $\mathcal{P}$ of S_1-F_1 is contained in the null space of F_4. Since S_1-F_1 is a semi-Fredholm operator, this implies that there exists an operator Y of finite rank in $\mathcal{L}(\mathcal{P})$ such that $Y(S_1-F_2) = F_4$ (see [8, Theorem 1]). We now apply a similarity transformation to (6) as follows:

$$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}\begin{pmatrix} U_1-F_1 & S_1-F_2 \\ B_1-F_3 & -F_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -Y & 1 \end{pmatrix},$$

obtaining a matrix of the form

$$\begin{pmatrix} Z & * \\ * & 0 \end{pmatrix}, \tag{7}$$

where $Z = U_1-F_1-(S_1-F_2)Y$. Since U_1 has infinite dimensional null space (no matter what k is) and since $F_1+(S_1-F_2)Y$ has finite rank, it is easily seen that Z has an infinite dimensional null space too. Hence Z is a commutator (this follows, for instance, from Theorem 5. 1), say $Z=[A, B]$. Consider now the two operator matrices

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & T \\ R & 0 \end{pmatrix}, \tag{8}$$

where R and T remain to be determined. Calculation shows that the commutator of the operators in (8) is the operator matrix

$$\begin{pmatrix} Z & (A-1)T \\ R(1-A) & 0 \end{pmatrix}. \tag{9}$$

Since A may be replaced by any translate $A+\lambda$ without changing any of these calculations, we may certainly arrange for $A-1$ to be invertible, whereupon it becomes a triviality to solve for R and T in (9) so as to make (9) equal to (7). Thus W_0 is similar to a commutator, and the theorem is proved. □

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