Extending mutually orthogonal partial latin squares

By CHARLES C. LINDNER in Auburn (Alabama, U.S.A.)

1. Introduction

By an $n\times n$ (partial) *latin square* is meant an $n\times n$ array such that (in some subset of the n^2 cells of the array) each of the cells is occupied by an integer from the set $\{1, 2, \dots, n\}$ and such that no integer from this set occurs in any row or column more than once. We will also refer to an $n\times n$ (partial) latin square as a *finite* (partial) latin square. By an *infinite* latin square is meant a countably infinite array of rows and columns such that each positive integer occurs exactly once in each row and column.

If P is a finite (partial) latin square we will denote by S_p the set of all cells which are occupied in P. If P and Q are (partial) latin squares of the same size, by (P, Q) is meant the set $\{(p_{ij}, q_{ij})\colon (i, j)\in S_P\cap S_Q\}$. If P and Q are finite (partial) latin squares and $|(P, Q)| = |S_P\cap S_Q|$ we say that P and Q are *orthogonal* and write $P\perp Q$. If P and Q are infinite latin squares we say that P and Q are orthogonal provided that $(P, Q) = Z\times Z$ (where Z is the set of all positive integers) and every pair of cells in different rows and columns are occupied by the same symbol in at most one of P and Q. As above if P and Q are orthogonal infinite latin squares we write $P\perp Q$.

In this paper the term latin square will mean either a finite or infinite latin square.

If $\{P_i\}_{i\in I}$ is a collection of mutually orthogonal latin squares of the same size we say that this collection is a *complete set* of mutually orthogonal latin squares provided that every pair of cells in different rows and columns are occupied by the same symbol in exactly one member of the collection. We note that if the latin squares in this collection are finite and based on $N=\{1, 2, \dots, n\}$ then $I=\{1, 2, \dots, n-1\}$. If the latin squares are infinite then I is the set of positive integers.

In this paper we prove the following theorem.

Theorem. *A finite collection of mutually orthogonal $n\times n$ partial latin squares can be embedded in a complete set of mutually orthogonal infinite latin squares.*

The following ideas are used in the proof.

By a *plane* we will always mean a set π which is the union of two disjoint sets $\mathscr{P}$ and $\mathscr{L}$ (the elements of which are called points and lines) and a relation I from $\mathscr{P}$ to $\mathscr{L}$ called *incidence*. If $(P, l) \in I$ we will say that the point P is on or belongs to the line l and that l contains P. If (P, l) and $(P, k) \in I$ we will say that the lines l and k intersect in the point P. With this convention we make the following definitions.

For the notion of a *partial plane, projective plane*, and *affine plane*, the reader is referred to [1].

If π_1 and π_2 are partial planes we say that π_1 is explicitly contained in π_2 and write $\pi_1 < \pi_2$ if and only if the following conditions are satisfied.

(i) The points and lines of π_1 are contained in π_2.

(ii) If the points P, Q and the line l are in π_1. and if P and Q belong to l in π_2 they belong to l in π_1.

(iii) If the lines l, k and the point P are in π_1 and the lines l and k intersect in P in π_2 they intersect in P in π_1.

2. Proof of the Theorem

Let $P_1, P_2, \ldots, P_t$ be a collection of mutually orthogonal $n \times n$ partial latin squares. We define a partial plane π_0 in which there are points P_{ij} $(i, j = 1, 2, \ldots, n)$ and lines l_{ij} $(i = 1, \ldots, t; j = 1, \ldots, n)$, where the point P_{rs} belongs to the line l_{ij} if and only if in P_i the cell (r, s) is occupied by j. We now successively define partial planes π_1, π_2, and π_3 so that $\pi_0 < \pi_1 < \pi_2 < \pi_3$ as follows.

The points of π_1 are the points of π_0 and the lines are those of π_0 along with the following lines. For each set of points $\{P_{i1}, P_{i2}, \ldots, P_{in}\}$ $(i = 1, 2, \ldots, n)$ we define a line h_i containing exactly these points. For each set of points $\{p_{1i}, P_{2i}, \ldots, P_{ni}\}$ $(i = 1, 2, \ldots, n)$ we define a line v_i containing exactly these points. For every pair of points not already belonging to one of the above lines we define a line containing exactly these two points.

The lines of π_2 are those in π_1 and the points are those in π_1 along with the following points. For the set of lines $\{h_1, \ldots, h_n\}$ define a point H belonging to exactly these lines. For the set of lines $\{v_1, \ldots, v_n\}$ define a point V belonging to exactly these lines. For each set of lines $\{l_{i1}, l_{i2}, \ldots, l_{in}\}$ $(i = 1, 2, \ldots, t)$ define a point L_i belonging to exactly these lines. For each pair of lines not intersecting in one of the above points define a point belonging to exactly these two lines.

The points of π_3 are those in π_2 and the lines of π_3 are the lines of π_2 along with the following lines. For the set of points $\{H, V, L_1, L_2, \ldots, L_t\}$ define a line p_∞ containing exactly these points. For every pair of points not contained in one of the above lines define a line containing exactly these two points.

From the definition of π_0, π_1, π_2, and π_3 it follows that $\pi_0 < \pi_1 < \pi_2 < \pi_3$. In [1] M. HALL has shown that if π is a partial plane there is a projective plane π' such that $\pi < \pi'$. In case π is finite, Hall's theorem leads to a countably infinite containing plane.

Let π be a countably infinite projective plane such that $\pi_3 < \pi$. Then $\pi_0 < \pi$. We now remove from π the line p_∞ along with the points belonging to this line to obtain an affine plane π^*. Among the points removed from π are the points $H, V, L_1, L_2, \dots, L_t$ so that in π^* the lines $h_1, \dots, h_n$; $v_1, \dots, v_n$; and $l_{i1}, \dots, l_{in}$ $(i=1, \dots, t)$ are parallel. Let $\mathscr{H}$ denote the pencil of lines in π^* containing the h's, $\mathscr{V}$ the pencil containing the v's, and $\mathscr{P}_i$ $(i=1, 2, \dots)$, the other pencils with the requirement that the lines $l_{i1}, \dots, l_{in}$ belong to $\mathscr{P}_i$. Label the lines in each pencil with the positive integers with the additional proviso that in $\mathscr{H}$ the line h_i is labeled i, in $\mathscr{V}$ the line v_i is labeled i, and in $\mathscr{P}_K$, $k=1, 2, \dots, t$ the line labeled l_{ki} is labeled i. Now construct a collection of infinite latin squares $C_1, C_2, \dots, C_i, \dots$ as follows. In C_k the cell (i, j) is occupied by x if and only if the line labeled x in $\mathscr{P}_K$ contains the point of intersection of the lines labeled i and j in $\mathscr{H}$ and $\mathscr{V}$ respectively. It is routine matter to check that the collection $C_1, C_2, \dots$ obtained in this manner is in fact a complete set of mutually orthogonal infinite latin squares and P_i is embedded in the upper left-hand corner of C_i $(i=1, 2, \dots, t)$.

This completes the proof of the theorem.

References

[1] M. HALL, JR., Projective planes, *Trans. Amer. Math. Soc.*, **54** (1943), 229—277.

[2] H. J. RYSER, *Combinatorial mathematics*, The Carus Mathematical Monographs, No. XIV, Math. Assoc. Amer. (1963).

AUBURN UNIVERSITY

(Received March 6, 1970)