On an extremum problem for polynomials

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Recently, P. TURÁN [8] treated the problem to determine lower bounds of the expression

$$M_n(p) = \inf_{\varrho \in P_{n-1}} \sup_{x \in [-1,+1]} |p(x)[x^n + \varrho(x)]|$$

for fixed but arbitrary values of the natural number n, where P_{n-1} is the set of polynomials of degree $n-1$ at most, and $p(x)$ is a given polynomial. In the present paper we consider the problem for arbitrary bounded functions $p(x) \geqq 0$; our estimates are sharper than those of TURÁN [8] and cover some of ELBERT's results [4], [5], too.

Theorem I. *For an arbitrary bounded* [1]) *function* $p(x) \geqq 0$

$$2^n M_n(p) \geqq G(p^*) \qquad (n = 1, 2, \dots) \tag{1}$$

and

$$\varlimsup_{n\to\infty} 2^n M_n(p) \leqq 2G(p^*), \tag{2}$$

where p^* *is the upper limit function of* p *and* [2, 3])

$$G(p^*) = \exp\left\{\frac{1}{\pi}\int_0^\pi \log p^*(\cos\theta)\, d\theta\right\}. \tag{3}$$

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[1]) If $p(x)$ is unbounded but $M_n(p)$ is finite for $n \geqq m$, then there exists a nonnegative polynomial of minimal degree $\pi_0(x) = x^m + \varrho(x)$ $(\varrho \in P_{m-1})$ for which $\pi_0 p$ is bounded. Clearly $M_n(p) = M_{n-m}(\pi_0 p)$ and we have[3]) $G(\pi_0 p^*) = G(\pi_0) G(p^*) = 2^{-m} G(p^*)$, so that (1) and (2) are valid even if $p(x)$ is unbounded.

[2]) The integral in (3) is defined, because p^* is bounded, positive and (as an upper limit function) semicontinuous from above, but it may take the value $-\infty$; in this case we set $G(p^*) = 0$.

[3]) If $p(x) = p^*(x) = |x-b_1|^{\beta_1} |x-b_2|^{\beta_2} \dots |x-b_k|^{\beta_k}$, where $b_1, b_2, \dots, b_k$ are arbitrary complex numbers, $\beta_1, \dots, \beta_k$ are real numbers and $\beta_i \geqq 0$ if $b_i \in [-1, +1]$, then we have

$$G(p^*) = 2^{-k} \prod_{j=1}^{k} |b_j + \sqrt{b_j^2 - 1}|$$

(see BERNSTEIN [1]); this is the case treated by TURÁN [8] and ELBERT [4], [5].

Proof of (1). We have[4] $M_n(p)=M_n(p^*)$. If $\log p^*(\cos\theta)\notin L$, (1) is satisfied in a trivial way, for its right hand side is zero. So we may assume $\log p^*(\cos\theta)\in L$.

For an arbitrary but fixed $\varepsilon>0$ we take a $\psi_n(x)=x^n+\cdots\in P_n$ for which

$$p^*(x)|\psi_n(x)|\leqq M_n(p^*)+\varepsilon = M_n(p)+\varepsilon. \tag{4}$$

By a well-known theorem of G. Szegő [7], the function

$$\varphi(z)=\exp\left\{\frac{1}{\pi}\int_0^\pi \frac{e^{i\theta}+z}{e^{i\theta}-z}\log p^*(\cos\theta)\,d\theta\right\}\qquad (|z|\leqq 1)$$

belongs to H^1 and satisfies $|\varphi(e^{i\theta})|=p^*(\cos\theta)$ a.e. Applying (4) to $x=\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ we find that

$$F(z)=2^n z^n\psi_n\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]\varphi(z)\in H^1$$

has for a.e. boundary values not exceeding $2^n[M_n(p^*)+\varepsilon]$ in modulus. As a consequence of $f\in H^1$ the maximum principle is applicable; so we obtain

$$G(p^*)=\varphi(0)=F(0)\leqq \operatorname{vrai}\max_\theta |F(e^{i\theta})|\leqq 2^n[M_n(p^*)+\varepsilon]=2^n[M_n(p)+\varepsilon]$$

and for $\varepsilon\to 0$ we get (1). Q.e.d.

Proof of (2). Since p^*, as an upper limit function, is bounded and semicontinuous from above, there exists a decreasing sequence $\{p_s(x)\}$ of nonvanishing continuous functions such that

$$\lim_{s\to\infty} p_s(x)=p^*(x)\qquad (x\in[-1,+1]).$$

Since $p^*(x)\leqq p_s(x)$, we have $M_n(p)=M_n(p^*)\leqq M_n(p_s)$, so that by a theorem of Bernstein [2]

$$\overline{\lim_{n\to\infty}}\, 2^n M_n(p)\leqq \lim_{n\to\infty} 2^n M_n(p_s)=2G(p_s).$$

Now, if $\log p^*(\cos\theta)\in L$, we obtain (2) from (3) by an application of Lebesgue's theorem on bounded convergence, taking $s\to\infty$. If $\log p^*(\cos\theta)\notin L$, we get from (3) by an indirect application of Fatou's lemma $\lim_{s\to\infty} G(p_s)=0$; this completes the proof of (2).

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[4]) Proof: For an arbitrary $\varepsilon>0$ there exists a $\varrho\in P_{n-1}$ such that $\sup_{x\in[-1,1]} p(x)|x^n+\varrho(x)|\leqq M_n(p)+\varepsilon$; we conclude that for every sequence $x_k\to x$ ($x_k\in[-1,+1]$) we have

$$\overline{\lim_{k\to\infty}}\,\{p(x_k)\,|x_k^n+\varphi(x_k)|\}\leqq M_n(p)+\varepsilon,$$

i.e. by continuity of $x^n+\varrho(x)$, $p^*(x)\,|x^n+\varrho(x)|\leqq M_n(p)+\varepsilon$ so that $M_n(p^*)\leqq M_n(p)+\varepsilon$. In turn, from $p\leqq p^*$ it follows $M_n(p)\leqq M_n(p^*)$, and these two results imply $M_n(p)=M_n(p^*)$.

Theorem II. *For an arbitrary function* $p(x) \geqq 0$ *and an arbitrary pair of natural numbers* $n<r$,

(5) $$2^n M_n(p) \geqq \frac{1}{2} 2^r M_r(p).$$

Conversely, for an arbitrary $\delta>0$ *and arbitrary natural number* n *there exists a continuous function* $s(x)=s(n, \delta; x)>0$ *such that*

(6) $$2^n M_n(s) < \frac{1+\delta}{2} 2^r M_r(s) = (1+\delta)G(s) \qquad (r = n+1, n+2, \ldots).$$

Proof of (6). The Chebyshev polynomial T_{r-n} satisfies $|T_{r-n}(x)| \leqq 1$ ($x \in [-1, +1]$) and has the leading coefficient 2^{r-n-1}. So we have

$$M_r(p) \leqq \inf_{\varrho \in P_{n-1}} \sup_{x \in [-1,+1]} |p(x) 2^{-r+n+1} T_{r-n}(x)[x^n + \varrho(x)]| = \\ = 2^{-r+n+1} M_n(p|T_{r-n}|) \leqq 2^{-r+n+1} M_n(p),$$

and multiplying by 2^{r-1} we get the desired inequality (4).

Proof of (6). Let $a>1$ and $s_a(x) = \left(1-\frac{x}{a}\right)^{-2n}$. By a result of BERNSTEIN ([3], pp. 11—14) we have $2^r M_r(s_a) = 2G(s_a)$ $(r = n+1, n+2, \ldots)$ and

$$2^n M_n(s_a) = \frac{2}{1+\left(a-\sqrt{a^2-1}\right)^{2n}} G(s_a).$$

To prove (6) we need only to observe that

$$\lim_{a \to 1+0} \frac{2}{1+\left(a-\sqrt{a^2-1}\right)^{2n}} = 1$$

and take $s=s_a$ for a sufficiently near to 1.

Theorem III. *For an arbitrary natural number* n *and arbitrary large* $A>0$ *there exists a continuous function* $p_A(x)>0$ *for which*

(7) $$2^n M_n(p_A) > A \lim_{r \to \infty} 2^r M_r(p_A).$$

Remark. This result is a consequence of an earlier theorem of ELBERT [5]. In the shorter proof what follows we make use of another idea of ELBERT, which is reproduced here with his permission.

Proof of Theorem III. Let $a=\frac{3}{2\sqrt{2}}>1$, $b=\frac{\sqrt{3}}{2}<1$, and $t(x)=\left(1-\frac{x}{a}\right)^m$, where m is a natural integer to be specified later. By Bernstein's theorem[3])

(8) $$\lim_{r \to \infty} 2^r M_r(t) = 2\left(\frac{a+\sqrt{a^2-1}}{2a}\right)^m = 2^{-m-1}\left(\frac{4}{3}\right)^m.$$

We have further by the transformation $x=b\xi$

$$M_n(t)=\min_{\varrho\in P_{n-1}}\max_{|x|\le 1}\left(1-\frac{x}{a}\right)^m|x^n+\varrho(x)|\ge\min_{\varrho\in P_{n-1}}\max_{|x|\le b}\left(1-\frac{x}{a}\right)^m|x^n+\varrho(x)|=$$

$$=\min_{\varrho^*\in P_{n-1}}\max_{|\xi|\le 1}\left(1-\frac{b}{a}\xi\right)^m|b^n\xi^n+\varrho^*(\xi)|=b^nM_n(t_b),$$

where

$$t_b(x)=\left(1-\frac{b}{a}x\right)^m.$$

Applying Theorem II and then Bernstein's theorem [3]) we obtain

$$2^nM_n(t)\ge b^n2^nM_n(t_b)\ge\frac{1}{2}b^n\lim_{r\to\infty}2^rM_r(t_b)=$$

$$=\frac{1}{2}b^n\left(\frac{\frac{a}{b}+\sqrt{\frac{a^2}{b^2}-1}}{2\frac{a}{b}}\right)^m=2^{-m-1}b^n\left(\frac{3}{2}\right)^m.$$

From (7) and (8) we get

$$\frac{2^nM_n(t)}{\lim\limits_{r\to\infty}2^rM_r(t)}\ge\frac{b^n}{4}\left(\frac{9}{8}\right)^m.$$

For a fixed value of n the right hand side exceeds by a suitable choice of m, any large $A>0$. Q.e.d.

References

[1] S. N. BERNSTEIN, Sur quelques propriétés asymptotiques des polynômes, *C. R. Acad. Sci. Paris*, **157** (1913) 1055—1057 (or *Collected Works*. I (Moscow, 1952), 207—208).

[2] S. N. BERNSTEIN, Sur les polynômes orthogonaux relatifs à un segment fini. II, *J. math. pures et appl.*, **10** (1931), 219—286 (or *Collected Vorks*. II (Moscow, 1954), 52—196).

[3] S. N. BERNSTEIN, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle* (Paris, 1926).

[4] Á. ELBERT, Über eine Vermutung von P. Erdős betreffs Polynome. II, *Studia Sci. Math. Hung.*, 3 (1969), 299—324.

[5] Á. ELBERT, An inequality for polynomials, *Annales Univ. Sci. Budapestiensis*, to appear.

[6] G. FREUD, *Orthogonale Polynome* (Budapest—Basel—Berlin, 1969).

[7] G. SZEGŐ, Über die Randwerte analytischer Funktionen, *Math. Annalen*, **84** (1921), 232—244.

[8] P. TURÁN, On an inequality of Čebyšev, *Annales Univ. Sci. Budapestiensis*, **11** (1968), 15—16.

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