S-objects in an abelian category

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1. Introduction

An abelian group G is an *S-group* if whenever K is a direct summand of G, then $G \cong G \oplus K$ [1]. G is an ID-*group* if G has an isomorphic proper direct summand [2]. In this paper we extend these concepts to an arbitrary abelian category with the emphasis on S-objects. Section 2 contains a few general properties of S-objects. In section 3 we investigate the relation of S-objects to ID-objects. We show that an ID-object in a C_3-category (i.e., satisfies the Grothendieck axiom A. B. 5) contains a non-zero S-object and we give a condition such that an S-object A in a complete C_3-category is isomorphic to an interdirect sum of countably many copies of A. In the last section we restrict our discussion to the category of abelian groups. We show several cases of a cancellation property for S-groups and conclude with the result that an abelian group whose torsion subgroup is an ID-group has a non-zero direct summand which is an S-group.

Throughout this paper **A** will denote an abelian category and A an arbitrary object in **A**. The word group will mean abelian group. Most of the notation is based on MITCHELL [6] with some taken from FUCHS [4] and the two main resource papers [1] and [2].

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2. S-objects

(2. 1) Definition. An object $A \in \mathbf{A}$ is an *S-object* if whenever B is a direct summand of A, then $A \cong A \oplus B$.

Theorem 2. 3, based on a similar result for direct sums of groups [1, Th. 3, p. 74] gives two large classes of S-objects.

(2. 2) Lemma. *Let* **A** *be complete (cocomplete). If* $A = \underset{i<\omega}{\times} A_i \left(\underset{i<\omega}{\oplus} A_i \right)$, *where* $A_i \cong A$ *for each i, then A is an* S-*object.*

Proof. Suppose $A = B \oplus L$. Then $A_i = B_i \oplus L_i$, where $B_i \cong B$ and $L_i \cong L$ for all i. Hence $A = \underset{i<\omega}{\times} A_i = \underset{i<\omega}{\times} (B_i \oplus L_i) \cong (\underset{i<\omega}{\times} B_i) \oplus (\underset{i<\omega}{\times} L_i) = B_0 \oplus (\underset{i<\omega}{\times} B_{i+1}) \oplus \oplus (\underset{i<\omega}{\times} L_i) \cong B_0 \oplus (\underset{i<\omega}{\times} (B_{i+1} \oplus L_i)) \cong B \oplus \underset{i<\omega}{\times} A_i = B \oplus A$. Therefore, A is an S-object.

Dually, A is an S-object if $A = \underset{i<\omega}{\oplus} A_i$.

(2. 3) Theorem. *Let* **A** *be complete (cocomplete). If* $A = \underset{\lambda \in \Lambda}{\times} B_\lambda \left(\underset{\lambda \in \Lambda}{\oplus} B_\lambda \right)$, *where* $|\Lambda| \geqq \aleph_0$ *and* $B_\lambda \cong B$ *for each* λ, *then A is an* S-*object.*

Proof. Partition the index set Λ into $\aleph_0$ disjoint subsets Λ_i such that $|\Lambda_i| = |\Lambda|$ for all i. Then $A = \underset{\lambda \in \Lambda}{\times} B_\lambda \cong \underset{i \in \omega}{\times} \left(\underset{i \in \Lambda_i}{\times} B_\lambda \right) = \underset{i<\omega}{\times} A_i$ where $A_i = \underset{i \in \Lambda_i}{\times} B_\lambda \cong A$ for each i. Therefore A is an S-object by Lemma 2. 2.

Dually, A is an S-object if $A = \underset{\lambda \in \Lambda}{\oplus} B_\lambda$.

Kaplansky [5, p. 12] raises three questions which he notes might be appropriate to consider for any specific structure of groups. It follows directly from the definition that test problems I and II are satisfied by S-objects in an arbitrary **A**.

(2. 4) Proposition. (Kaplansky's test problems I and II.) *Let A and B be* S-*objects in* **A** *then:* I. *A isomorphic to a direct summand of B and B isomorphic to a direct summand of A implies* $A \cong B$, *and* II. $A \oplus A \cong B \oplus B$ *implies* $A \cong B$.

For an S-object A, it is obvious that $A \cong \underset{n}{\oplus} A$ for any $n < \omega$ since $A \cong A \oplus A$. However, $A \not\cong \underset{\aleph_0}{\oplus} A$ in general as the following example shows.

(2. 5) Example. Let $P = \underset{\aleph_0}{\times} Z$ where Z is the additive group of the integers. Then P is an S-group by Theorem 2. 3 and $\underset{\aleph_0}{\oplus} P \cong \underset{\aleph_0}{\oplus} (Z \oplus P) \cong (\underset{\aleph_0}{\oplus} Z) \oplus (\underset{\aleph_0}{\oplus} P)$. Nunke [7, Th. 5, p. 69] shows that every direct summand of a product of copies of Z is a product of copies of Z. Thus $P \not\cong \underset{\aleph_0}{\oplus} P$.

3. ID-objects

Many of the results in this section are extensions and applications to S-objects of the results and techniques in [2].

(3. 1) Definition. An object $A \in$ **A** is called an ID-*object* if A has an isomorphic proper direct summand.

(3. 2) Lemma. *If $A \neq 0$ is an S-object, then A is an ID-object.*

(3. 3) Lemma. *An object $A \in \mathbf{A}$ is an ID-object if and only if there exist $\varphi, \psi \in [A, A]$ such that $\psi\varphi = 1_A$ and $\varphi\psi \neq 1_A$.* ($[A, A]$ is the set of all morphisms from A to A in $\mathbf{A}$.)

Proof. Let A be an ID-object, then $A = B \oplus L$, $L \neq 0$, and there is an isomorphism $\varphi_1 : A \rightarrowtail\!\!\twoheadrightarrow B$. Let $\varphi = u_B \varphi_1$ where u_B is the injection of B into the coproduct. Let $\psi : B \oplus L \to A$ be the unique map defined by the definition of coproduct such that $\psi u_B = \varphi_1^{-1}$ and $\psi u_L = 0$. Then $\psi\varphi = \psi u_B \varphi_1 = \varphi_1^{-1}\varphi_1 = 1_A$ and $\varphi\psi(A) = B$ so $\varphi\psi \neq 1_A$.

Conversely, if $\psi\varphi = 1_A$, then φ is a monomorphism and the exact sequence $0 \to A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} A \to A/\varphi(A) \to 0$ splits so that $A = \varphi(A) \oplus A/\varphi(A) = \varphi(A) \oplus \operatorname{Ker} \psi$ [6, Prop. 19. 1*, p. 32]. But $\varphi\psi \neq 1_A$ implies $\operatorname{Ker} \psi \neq 0$. Therefore, A is isomorphic to a proper direct summand $\varphi(A)$.

Thus, ID-objects can be studied by means of the following definition.

(3. 4) Definition. An *ID-system* is a triple $\langle A; \varphi, \psi \rangle$ where $A \in \mathbf{A}$ and $\varphi, \psi \in [A, A]$ such that $\psi\varphi = 1_A$.

Since any S-object A is an ID-object it determines an ID-system. An S-object actually determines a set of distinct ID-systems. This is shown in the following characterization of S-objects.

(3. 5) Proposition. *Let $\mathbf{B}$ be a representative set of non-isomorphic direct summands of A. A is an S-object if and only if there exists a set $\{(\varphi_B, \psi_B) : B \in \mathbf{B}\} \subset \subset [A, A] \times [A, A]$ such that $\psi_B \varphi_B = 1_A$ and $\operatorname{Ker} \psi_B \cong B$ for all $B \in \mathbf{B}$.*

Proof. We need to show first that $\mathbf{B}$ is a set. If B is a direct summand of A, then the projection onto B followed by the injection of B into A is a morphism $\gamma_B \in [A, A]$ such that $\gamma_B(A) \cong B$. Thus if $C \ncong B$ as subobjects, $\gamma_B(A) \ncong \gamma_C(A)$ so $\gamma_B \neq \gamma_C$. Therefore, $\mathbf{B}$ is in one-to-one correspondence with a class of distinct morphisms in $[A, A]$. Since $[A, A]$ is a set, $\mathbf{B}$ is a set.

If A is an S-object and $A = B \oplus M$, $B \in \mathbf{B}$, then there is an isomorphism $\alpha : A \oplus B \rightarrowtail\!\!\twoheadrightarrow A$. Let $u : A \to A \oplus B$ be the injection of A into the coproduct and p the projection onto A. Define $\varphi_B = \alpha u$ and $\psi_B = p\alpha^{-1}$. Then $\psi_B \varphi_B = p\alpha^{-1}\alpha u = pu = 1_A$ and $\operatorname{Ker} \psi_B \cong \operatorname{Ker} p = B$.

Conversely, let $A = B' \oplus M$ and $B \in \mathbf{B}$ such that $B \cong B'$ as subobjects of A. $\psi_B \varphi_B = 1_A$, so φ_B is monic and $A = \varphi_B(A) \oplus \operatorname{Ker} \psi_B$ as in Lemma 3. 3. But $\varphi_B(A) \cong A$ and $\operatorname{Ker} \psi_B \cong B \cong B'$ so $A \cong A \oplus B'$. Therefore, A is an S-object.

(3. 6) Theorem. *Let $\mathbf{A}$ be C_3, A an ID-object in $\mathbf{A}$, then A contains a non-zero S-object.*

Proof. Since A is an ID-object, there is an ID-system $\langle A; \varphi, \psi\rangle$ for A such that $\operatorname{Ker}\psi \neq 0$. Let $H = \operatorname{Ker}\psi$. Then by repeatedly applying φ to A, A splits as $A = H \oplus \varphi(A) = H \oplus \varphi(H) \oplus \varphi^2(A) = \cdots$, where $\varphi^n(A) \cong A$ and $\varphi^n(H) \cong H$ for all $n < \omega$ $(\varphi^0(H) = H)$. Then $\{\varphi^n(H) : n < \omega\}$ is a set of subobjects of A such that $\bigoplus_{n=0}^{m} \varphi^n(H)$ is a direct summand of A for every $m < \omega$. Clearly $\left\{\bigoplus_{n=0}^{m} \varphi^n(H) : m < \omega\right\}$ is a direct system and $\varinjlim_{m<\omega} \left(\bigoplus_{n=0}^{m} \varphi^n(H)\right) = \bigoplus_{n<\omega} \varphi^n(H)$ (see [6, p. 48, Example 1]). But $\varinjlim_{m<\omega} \left(\bigoplus_{n=0}^{m} \varphi^n(H)\right) = \bigcup_{n<\omega} \varphi^n(H) \subset A$ by [6, Prop. 1. 2, p. 82] since $\mathbf{A}$ is C_3. Therefore, $\bigoplus_{n<\omega} \varphi^n(H)$ is a subobject of A and by Theorem 2. 3 it is an S-object.

(3. 7) Corollary. *Let* $\mathbf{A}$ *be* C_3, A *an* S-*object. Then* A *contains an* S-*object isomorphic to* $\bigoplus_{\aleph_0} A$.

Proof. Since $A \cong A \oplus A$, let $\langle A; \varphi, \psi\rangle$ be an ID-system for A such that $H = \operatorname{Ker}\psi \cong A$. Then $\varphi^n(H) \cong H \cong A$ and $\bigoplus_{n<\omega} \varphi^n(H) \cong \bigoplus_{\aleph_0} A$ so the results follows from Theorem 3. 6 and its proof.

By imposing additional hypotheses we are able to extend the conclusion in 3. 7 such that an S-object is isomorphic to an interdirect sum of countably many copies of itself. In Theorem 3. 9 we let $\varphi^\omega A = \bigcap_{n<\omega} \varphi^n(A)$. This intersection exists since we assume A to be complete.

(3. 8) Definition. Let $\mathbf{A}$ be complete C_3 with $\{A_i : i \in I\} \subset \mathbf{A}$. An object $A \in \mathbf{A}$ is called an *interdirect sum* of the A_i if

$$\bigoplus_{i\in I} A_i \subset A \subset \bigtimes_{i\in I} A_i$$

(3. 9) Theorem. *Let* $\mathbf{A}$ *be complete* C_3. *If* A *is an* S-*object with* ID-*system* $\langle A; \varphi, \psi\rangle$ *where* $\operatorname{Ker}\psi \cong A$ *and if* $\varphi^\omega A$ *is a direct summand of* A, *then* A *is isomorphic to an interdirect sum of countably many copies of* A.

Proof. Let $H = \operatorname{Ker}\psi$ and $K = \varphi^\omega A$. From the proof of Theorem 3. 6 we have $\bigoplus_{n<\omega} \varphi^n(H) \subset A$ and

$$(*) \qquad A = H \oplus \varphi(H) \oplus \cdots \oplus \varphi^n(H) \oplus \varphi^{n+1}(A).$$

Thus let $\alpha_n : A \to \varphi^n(H)$ be the projection defined by $(*)$ and let $p_n : \bigtimes_{n<\omega} \varphi^n(H) \to \varphi^n(H)$ be the projection from the product. Then by the definition of product there exists a unique $\alpha : A \to \bigtimes_{n<\omega} \varphi^n(H)$ such that $p_n\alpha = \alpha_n$ for all $n < \omega$. Let $L = \operatorname{Im}\alpha$.

Now from $(*)$ we see that $\operatorname{Ker}\alpha_n = H \oplus \cdots \oplus \varphi^{n-1}(H) \oplus \varphi^{n+1}(A)$.

Thus $\bigcap_{n=0}^{m} \operatorname{Ker}\alpha_n = \varphi^{m+1}(A)$, and $\bigcap_{n<\omega} \operatorname{Ker}\alpha_n = \bigcap_{n<\omega} \varphi^n(A) = \varphi^\omega A$.

Thus, by an exercise in MITCHELL [6, Ex. 8, p. 37] $\operatorname{Ker}\alpha=\varphi^{\omega}A=K$. Since $\varphi^{\omega}A$ is a direct summand by hypothesis, $A = K\oplus L$.

Claim $A\cong L$. $A = H\oplus\varphi(A)$ by (*) and $K\subset\varphi(A)$ by definition. Thus, by the modular law [3, p. 103, Exercise A] $\varphi(A) = A\cap\varphi(A) = K\oplus[L\cap\varphi(A)]$ implies $A = H\oplus K\oplus[L\cap\varphi(A)]$ so that $L\cong A/K\cong H\oplus[L\cap\varphi(A)]$. Now $H\cong A$ and A an S-object implies $H\oplus K\cong A\oplus K\cong A\cong H$. Hence $A = K\oplus H\oplus[L\cap\varphi(A)]\cong$ $\cong H\oplus[L\cap\varphi(A)]\cong L$.

Finally, we need to show that $\bigoplus_{n<\omega}\varphi^n(H)\subset L$. Let β_n and γ_n be the injection of $\varphi^n(H)$ into A and $\bigtimes_{n<\omega}\varphi^n(H)$ respectively. Then $\alpha\beta_n=\gamma_n$ since $p_j\alpha\beta_n=\alpha_j\beta_n$ is the identity on $\varphi^n(H)$ if $j=n$ and is 0 if $j\neq n$ and similary for $p_j\gamma_n$. Thus α restricted to $\bigoplus_{n<\omega}\varphi^n(H)$ is the natural map $\delta\colon\bigoplus_{n<\omega}\varphi^n(H)\to\bigtimes_{n<\omega}\varphi^n(H)$. By hypothesis and [6, Cor. 1. 3, p. 83], **A** is C_2, thus δ is a monomorphism. Since α factors through L we have $\bigoplus_{n<\omega}\varphi^n(H)\subset L\subset\bigtimes_{n<\omega}\varphi^n(H)$.

Since $\varphi^n(H)\cong A$ for all $n<\omega$, L is isomorphic to an interdirect sum of countably many copies of A. Since $A\cong L$, the proof is complete.

4. Applications to abelian groups

In this section we restrict our attention to the category of abelian groups. We start with a cancellation property for S-groups. This follows the standard pattern of considering the reduced and divisible cases separately.

(4. 1) Proposition. *Suppose G is an S-group, $G = K\oplus L$, K finitely generated, then $G\cong L$.*

Proof. G an S-group implies $G\cong K\oplus G$ so that $K\oplus G\cong K\oplus L$. Thus $G\cong L$ by [8, Cor. 8, p. 900].

(4. 2) Theorem. *Let G be a reduced p-group, G an S-group, and $G = K\oplus L$ where K contains no non-zero S-group, then $G\cong L$.*

Proof. Suppose K is infinite and let B be a basic subgroup for K. Then K infinite implies $|B|=m\geqq\aleph_0$ so that $B[p]\cong\bigoplus_m C(p)$ is an S-group. Thus K is finite and therefore finitely generated. By Proposition 4. 1, $G\cong L$.

(4. 3) Corollary. *Let T be a reduced torsion group, T an S-group, and $T = K\oplus L$ where K contains no non-zero S-group, then $T\cong L$.*

Proof. $T = \bigoplus_{p\in\pi} T_p$ and each T_p is an S-group [1, Cor. 2, p. 72]. Also $T = K\oplus L$ implies $T_p = K_p\oplus L_p$ and K_p contains no non-zero S-group since K contains no non-zero S-group. Thus Theorem 4. 2 implies $T_p\cong L_p$ and so $T\cong L$.

The conditions in Theorem 4. 2 are not sufficient to guarantee that G is an S-group. That is, the following is an example of a group G such that if $G = K \oplus L$ and K contains no non-zero S-groups, then $G \cong L$, however, G is not an S-group.

(4. 4) Example. By ZIPPIN [9, p. 98—99], there is a reduced countable p-group G such that $f(G, n) = \aleph_0$, $n < \omega$, and $f(G, \omega) = 1$ where $f(G, n)$ is the n^{th} Ulm invariant of G. If $G = K \oplus L$ where K contains no non-zero S-group, then, as in 4. 2, K is finite so that $f(K, n)$ is finite for $n < \omega$ and $f(K, \omega) = 0$. By the properties of Ulm invariants and by ULM's theorem [5, p. 27], it follows that $G \cong L$. However, G is not an S-group since $f(G, \omega) = 1$ [1, Th. 2, p. 73].

(4. 5) Theorem. *Let D be a divisible group, D an* S-*group, and $D = K \oplus L$ where K contains no non-zero* S-*groups, then $D \cong L$.*

Proof. By [1, Th. 2, p. 73] the torsion free rank of D is zero or infinite and the p-rank of D is zero or infinite for each $p \in \pi$. Now K is also divisible and if its torsion free rank were infinite or if its p-rank were infinite for any p, K would contain an S-group by Theorem 2. 3 (or [1, Th. 3, p. 74]). Thus, the torsion free rank of L and the p-rank of L for each p must be the same as the corresponding rank of D. Therefore, $D \cong L$.

We can now prove the general torsion case by splitting the group into its divisible and reduced components and applying 4. 3 and 4. 5. We also need the fact that a group is an S-group if and only if its reduced and divisible components are both S-groups [1, Cor. 1, p. 72].

(4. 6) Theorem. *Let T be a torsion group, T an* S-*group, and $T = K \oplus L$ where K contains no non-zero* S-*groups, then $T \cong L$.*

We next note that for groups Theorem 3. 9 has a special interpretation [see 2].

(4. 7) Proposition. *If G is an* S-*group with* ID-*system $\langle G; \varphi, \psi \rangle$ where* Ker $\psi \cong G$ *and if $\varphi^{\omega} G$ is a direct summand of G, then G is isomorphic to a total shift invariant subgroup of $\underset{\aleph_0}{\times} G$.*

The following gives a more involved example than Theorem 2. 3 of an S-group and demonstrates a simple application of Proposition 4. 7 (and thus of Theorem 3. 9).

(4. 8) Example. Let $P = \underset{\aleph_0}{\times} Z$ and $F = \underset{\aleph_0}{\oplus} Z$ where Z is the additive group of the integers. P and F are both S-groups by Theorem 2. 3. We will show that $P \oplus F$ is also an S-group.

Suppose $P \oplus F = A \oplus B$. Let φ be the projection of $P \oplus F$ onto F. Letting φ_A be the restriction of φ to A we get the exact sequence $0 \to \operatorname{Ker} \varphi_A \to A \xrightarrow{\varphi_A} F$ where $\varphi_A(A)$ is free since it is a subgroup of a free group. Since $A / \operatorname{Ker} \varphi_A \cong \varphi_A(A)$, we have $A = \operatorname{Ker} \varphi_A \oplus L$, where L is free [4, Th. 9. 2, p. 38]. Clearly L has countable rank. Now, Ker $\varphi_4 = A \cap P$ and $A \cap P$ is a direct summand of $P \oplus F$ since $P \oplus F = A \oplus B = A \cap P \oplus L \oplus B$. Thus $A \cap P \subset P$ implies $P = A \cap P \oplus P \cap (L \oplus B)$ by the modular law. So $A = A \cap P \oplus L$ where $A \cap P$ is a direct summand of P and hence a product of copies of Z [7, Th. 5, p. 69]. Therefore, $A \cong \bigoplus_n Z$, $A \cong P$, $A \cong F$, or $A \cong P \oplus F$. In any case, $P \oplus F \oplus A \cong P \oplus F$ so $P \oplus F$ is an S-group.

Let $G = (\bigoplus_{i<\omega} Z_i) \oplus (\times_{j<\omega} Z'_j)$ where $Z_i \cong Z \cong Z'_j$ for all i and j. Then $G \cong P \oplus F$ and so is an S-group. It is obvious that G is an interdirect sum of countably many copies of Z. With 4. 7 we can also show that G is isomorphic to a total shift invariant subgroup of $\times_{\aleph_0} G$. Define $\varphi: G \to G$ by $\varphi(Z_i) = Z_{2i}$ and $\varphi(Z'_j) = Z'_{2j}$, then

$$G = [(\bigoplus_{i<\omega} Z_{2i}(\oplus) \times_{j<\omega} Z'_{2j})] \oplus [(\bigoplus_{i<\omega} Z_{2i-j}) \oplus (\times_{j<\omega} Z'_{2j-1})] = \varphi(G) \oplus H$$

and $\varphi^n(G) = (\bigoplus_{i<\omega} Z_{2^n i}) \oplus (\times_{j<\omega} Z'_{2^n j})$ so $\varphi^\omega G = 0$ and is thus a direct summand of G and Proposition 4. 7 applies.

Clearly if an object A has a direct summand which is an S-object, A is an ID-object. We conclude with the converse for torsion groups. 4. 10 may also be considered a special case of Theorem 3. 6.

(4. 9) Lemma. *If a reduced p-group G is an* ID-*group, then G has a non-zero direct summand which is a bounded* S-*group.*

Proof. By [2, Th. 2. 9, p. 23], G an ID-group implies $f(G, n)$ is infinite for some integer n. Thus $B_n = \bigoplus_{f(G,n)} C(p^n)$ is an S-group (B_n is the n^{th} component of a basic subgroup for G). But B_n is bounded and is a direct summand of G.

(4. 10) Theorem. *If G_T is an* ID-*group, then G has a non-zero direct summand which is an* S-*group.* (G_T is the maximum torsion subgroup of G.)

Proof. Since G_T is an ID-group, by [2, Th. 2. 6, p. 23] $(G_T)_p$ is an ID-group for some p. Let $(G_T)_p = D_p \oplus R_p$ where D_p is divisible and R_p is reduced. Then, by [2, Th. 2.8, p. 23], D_p or R_p is an ID-group. If D_p is an ID-group, D_p has infinite p-rank and is thus an S-group by [1, Th. 3, p. 74]. D_p is also a direct summand of G since it is divisible. If R_p is an ID-group, then by Lemma 4. 9, R_p has a non-zero direct summand K which is a bounded S-group. Since R_p is pure in G, K is also pure in G and is a direct summand of G by [5, Th. 7, p. 18].

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