# On some inequalities concerning series of positive terms

By LÁSZLÓ LEINDLER in Szeged

Recently A. PRÉKOPA [1] has proved the integral inequality

$$\int_{-\infty}^{\infty} \sup_{x+y=t} f(x)g(y)\,dt \geqq 2\Big(\int_{-\infty}^{\infty} f^2(x)\,dx\Big)^{1/2}\Big(\int_{-\infty}^{\infty} g^2(y)\,dy\Big)^{1/2}, \tag{1}$$

where $f(x)$ and $g(y)$ are arbitrary non-negative measurable functions.

It seems worth while to observe that the formal analogue

$$\sum_{n=-\infty}^{\infty} \sup_{k+l=n} a_k b_l \geqq 2\left(\sum_{k=-\infty}^{\infty} a_n^2\right)^{1/2}\left(\sum_{l=-\infty}^{\infty} b_l^2\right)^{1/2}$$

of the inequality (1) is in general not true. (See e. g. the sequences $a_0=b_0=1$ and $a_n=b_n=0$ if $n\neq 0$.) But we will show that without the factor 2 the inequality does hold, i. e. we have

$$\sum_{n=-\infty}^{\infty} \sup_{k+l=n} a_k b_l \geqq \left(\sum_{k=-\infty}^{\infty} a_k^2\right)^{1/2}\left(\sum_{l=-\infty}^{\infty} b_l^2\right)^{1/2} \tag{2}$$

for any non-negative sequences $\{a_n\}$ and $\{b_n\}$.

First we give a very short and simple proof of (2). Next we generalize (2) as follows:

Theorem. *Suppose that* $1\leqq r, s\leqq\infty$ *and* $\frac{1}{r}+\frac{1}{s}=1+\frac{1}{\gamma}$, *where* $1\leqq\gamma\leqq\infty$. *Then for non-negative* $a_n, b_n$ *we have*

$$\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} a_k^\gamma b_{n-k}^\gamma\right)^{1/\gamma} \geqq \left(\sum_{k=-\infty}^{\infty} a_k^r\right)^{1/r}\left(\sum_{n=-\infty}^{\infty} b_n^s\right)^{1/s}. \;*) \tag{3}$$

---

*) If $c_k\geqq 0$ and $\gamma=\infty$, then $\left\{\sum_{k=-\infty}^{\infty} c_k^\gamma\right\}^{1/\gamma}$ means $\sup_k c_k$.

Let us formulate the special case $\gamma=\infty$ of (3) as a

Corollary. *If* $p, q\geqq 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\sum_{n=-\infty}^{\infty} \sup_k a_k b_{n-k} \geqq \left(\sum_{n=-\infty}^{\infty} a_n^p\right)^{1/p} \left(\sum_{n=-\infty}^{\infty} b_n^q\right)^{1/q}. \tag{4}$$

Our theorem can be generalized from two to any finite number of series with a straightforward generalization of the proof which follows.

E.g. *we have*

$$\sum_{n=-\infty}^{\infty}\left[\sum_{i+j+k=n}(a_i b_j c_k)^{\gamma}\right]^{1/\gamma} \geqq \left[\sum_{i=-\infty}^{\infty} a_i^r\right]^{1/r}\left[\sum_{j=-\infty}^{\infty} b_j^s\right]^{1/s}\left[\sum_{k=-\infty}^{\infty} c_k^t\right]^{1/t},$$

*where* $a_n, b_n, c_n$ *are non-negative and* $1\leqq r, s, t\leqq\infty$; $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=2+\frac{1}{\gamma}\,(1\leqq\gamma\leqq\infty)$.

The integral analogue of our result will be published in another paper. The method of proof to be given there is quite different from that of Prékopa and also slightly different from that one given for series in the present paper.

Proof of (2). Denote

$$\|a_n\| = \left(\sum_{n=-\infty}^{\infty} a_n^2\right)^{1/2}, \quad \|b_n\| = \left(\sum_{n=-\infty}^{\infty} b_n^2\right)^{1/2} \quad \text{and} \quad c_n = \sup_k a_k b_{n-k}.$$

We may assume that $0<\Sigma c_n<\infty$. This assumption implies that not all $a_n$ and $b_n$ vanish, and $0<\|a_n\|<\infty$ and $0<\|b_n\|<\infty$, indeed, for any $k$ and $n$, $c_n\geqq a_k b_{n-k}$. Taking the Cauchy product of the series $\Sigma a_n^2$ and $\Sigma b_n^2$ we obtain

$$\|a_n\|^2\|b_n\|^2 = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_k^2 b_{n-k}^2 \leqq \sum_{n=-\infty}^{\infty} c_n \sum_{k=-\infty}^{\infty} a_k b_{n-k} \leqq \sum_{n=-\infty}^{\infty} c_n \|a_n\|\cdot\|b_n\|.$$

Hence (2) follows evidently.

Proof of (3). We may also assume that the sum on the left-hand side of (3) has finite value and that not all $a_n$ and $b_n$ vanish. If $a_\mu>0$, then we have

$$\left[\sum_{k=-\infty}^{\infty}(a_k b_{n-k})^{\gamma}\right]^{1/\gamma} \geqq a_\mu b_{n-\mu} \quad \text{for any} \quad n,$$

and hence

$$\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}(a_k b_{n-k})^{\gamma}\right]^{1/\gamma} \geqq a_\mu \sum_{n=-\infty}^{\infty} b_n.$$

Therefore $\Sigma b_n < \infty$. For analogous reasons, $\Sigma a_n < \infty$. Since $r \geqq 1$ and $s \geqq 1$ we also have

(5) $$A \equiv \left[\sum_{n=-\infty}^{\infty} a_n^r\right]^{1/r} < \infty \quad \text{and} \quad B \equiv \left[\sum_{n=-\infty}^{\infty} b_n^s\right]^{1/s} < \infty.$$

If $r=\infty$ or $s=\infty$, then $\gamma=\infty$ and $s=1$ or $r=1$, respectively. In these cases (3) holds. If e.g. $r=\infty$, then by (5) there exists $v$ such that $a_v = \sup a_k$; thus the inequality

$$a_v b_n \leqq \sup_k a_k b_{n+v-k}$$

holds for all $n$; hence we obtain that

$$\sum_{n=-\infty}^{\infty} a_v b_n \leqq \sum_{n=-\infty}^{\infty} \sup_k a_k b_{n+v-k}$$

and this is what was to be proved.

If both $r$ and $s$ are finite we set

(6) $$c_n = a_n/A \quad \text{and} \quad d_n = b_n/B,$$

and we have

(7) $$\sum_{n=-\infty}^{\infty} c_n^r = 1 \quad \text{and} \quad \sum_{n=-\infty}^{\infty} d_n^s = 1.$$

Taking the Cauchy product of these two series we obtain

(8) $$\sum_{n=-\infty}^{\infty} I_n = 1, \quad \text{where} \quad I_n = \sum_{k=-\infty}^{\infty} c_k^r d_{n-k}^s.$$

Next we prove that

(9) $$I_n \leqq \left[\sum_{k=-\infty}^{\infty} (c_k d_{n-k})^\gamma\right]^{1/\gamma}.$$

If $\gamma=1$, then $r=s=1$, and thus in (9) equality holds. If $1<\gamma\leqq\infty$, set $\gamma' = \frac{\gamma}{\gamma-1}$ $(1 \leqq \gamma' < \infty)$. Then by the inequality of Hölder,

$$I_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k} c_k^{r-1} d_{n-k}^{s-1} \leqq J_n \left[\sum_{k=-\infty}^{\infty} (c_k d_{n-k})^\gamma\right]^{1/\gamma},$$

where

$$J_n = \left[\sum_{k=-\infty}^{\infty} c_k^{(r-1)\gamma'} d_{n-k}^{(s-1)\gamma'}\right]^{1/\gamma'},$$

thus we have to prove that $J_n \leqq 1$. If $r=1$ then $(s-1)\gamma' = s$, and if $s=1$ then

$(r-1)\gamma' = r$; thus in these cases $J_n \leqq 1$ follows immediately from (7). If both $r$ and $s$ are greater than 1, we can use (7) through the inequality

$$xy \leqq \frac{x^p}{p} + \frac{y^q}{q} \qquad (x, y \geqq 0),$$

applied to $x = c_k^{(r-1)\gamma'}$, $y = d_{n-k}^{(s-1)\gamma'}$, $p = \dfrac{r}{(r-1)\gamma'}$ and $q = \dfrac{s}{(s-1)\gamma'}$ (note that $\dfrac{1}{p} + \dfrac{1}{q} = 1$); then we get

$$J_n^{\gamma'} \leqq \sum_{k=-\infty}^{\infty} \left( \frac{1}{p} c_k^r + \frac{1}{q} d_{n-k}^s \right) = 1.$$

Now by (5), (7), (8) and (9) we have

$$1 = \sum_{n=-\infty}^{\infty} I_n \leqq \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} \left(\frac{a_k}{A}\right)^{\gamma} \left(\frac{b_{n-k}}{B}\right)^{\gamma} \right]^{1/\gamma} = \frac{1}{A \cdot B} \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} (a_k b_{n-k})^{\gamma} \right]^{1/\gamma},$$

which implies (3).

The proof is now complete.

My grateful acknowledgement is due to Professor BÉLA SZŐKEFALVI-NAGY for stimulating conversations.

## Reference

[1] A. PRÉKOPA, Logarithmic concave measures with application to stochastic programming, *Acta Sci. Math.*, **32** (1971), 301—316.

*(Received January 28, 1971)*