Uniform embedding of a metric space in Hilbert space

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It is known that every regular space having a σ-discrete base (or, what is the same, having a σ-locally finite base) can be homeomorphically embedded in a Hilbert space (SMIRNOV [1]). In the following we prove that if, in addition, in a regular space a particular metric is given, the embedding can be chosen uniformly continuous. Thus we shall prove the following

Theorem. *Every metric space X can be embedded in a Hilbert space by a uniformly continuous homeomorphism.*

Proof. First we give a definition: A collection of subsets of a space is called *discrete* if every point of the space has a neighbourhood which intersects at most one element of the collection. The collection is called *σ-discrete* if it can be decomposed into an at most countable infinity of discrete subcollections.

The proof of the theorem will be based on the result of BING [2] according to which *every metrizable space has a σ-discrete base.*

Thus, let $\mathscr{B}=\bigcup_{1}^{\infty}\mathscr{B}_n$ be a σ-discrete base of X, where the subcollections $\mathscr{B}_n$ $(n=1, 2, \dots)$ are discrete. We may also assume that every element of $\mathscr{B}$ has a diameter less than 1. For fixed $U\in\mathscr{B}$ and natural number j we consider the set

$$U^j=\left\{x:\varrho(x,\overline{U})>\frac{1}{j}\right\},$$

where $\overline{U}$ denotes the complement of U, and the function

$$f_{(U,j)}(x)=\frac{\varrho(x,\overline{U})}{\varrho(x,\overline{U})+\varrho(x,U^j)}$$

(ϱ denotes the metric of X). Thus we have $0\leqq f_{(U,j)}(x)\leqq 1$; and as for fixed $x\in X$ and for every natural number i there exists at most one $U\in\mathscr{B}_i$ with $x\in U$, it follows that $f_{(U,j)}(x)=0$ for every pair (U,j) with the possible exception of at most a countable infinity of pairs (U,j).

Let $k(i,j)$ be a 1—1 mapping which maps the set of all pairs onto the set of natural numbers and define

$$g_{(U,j)}(x)=f_{(U,j)}(x)/k(i,j)$$

for $U\in\mathscr{B}_i$. Let H be a Hilbert space the dimension of which is equal to the cardinality of all possible pairs (U,j) and for any $x\in X$ consider the function

$$G(x)=\{g_{(U,j)}(x)\}_{(U,j)}$$

which evidently maps X into H. We show that $G(x)$ is a homeomorphism which maps X into H in a uniformly continuous manner. To do this first we remark that both $\varrho(x,\overline{U})$ and $\varrho(x,U^j)$ fulfil a Lipschitz condition with constant 1, further $\varrho(x,\overline{U})\leqq 1$ and $\varrho(x,\overline{U})+\varrho(x,U^j)\geqq\frac{1}{j}$. Thus a simple calculation shows that if $\varrho(x,y)\leqq d$ then

$$|f_{(U,j)}(x)-f_{(U,j)}(y)|\leqq(2j^2+j)d.$$

But for fixed $x,y\in X$ and for every pair i,j there are among the numbers $\{f_{(U,j)}(x),f_{(U,j)}(y)\}_{U\in\mathscr{B}_i}$ at most two different from zero, thus we get for any natural number n

$$\|G(x)-G(y)\|^2=\sum_{(U,j)}|g_{(U,j)}(x)-g_{(U,j)}(y)|^2\leqq 2d^2\sum_{k(i,j)\leqq n}\frac{(2j^2+j)^2}{k^2(i,j)}+2\sum_{m=n}^{\infty}\frac{2}{m^2}.$$

Let $\varepsilon>0$ be given and choose n so large that $\sum_{m=n}^{\infty}\frac{1}{m^2}<\frac{\varepsilon^2}{4}$ and d so small that

$$d^2\sum_{k(i,j)\leqq n}\frac{(2j^2+j)^2}{k^2(i,j)}<\frac{\varepsilon^2}{4};$$

then $\|G(x)-G(y)\|<\varepsilon$, which gives the uniform continuity of $G(x)$.

On the other hand if, $x\neq y$ then there exist U and i such that $x\in U\in\mathscr{B}_i$ and $y\in\overline{U}$. If j is large enough then also $x\in U^j$. But in this case $f_{(U,j)}(x)=1$ and $f_{(U,j)}(y)=0$ which shows that G^{-1} exists.

The continuity of G^{-1} can be proved in the following manner. Let V be an arbitrary neighbourhood of $x\in X$. Then there exists i,j and $U\subset V$ with $x\in U^j\subset\subset U\in\mathscr{B}_i$; now if $\|G(x)-G(y)\|<\frac{1}{k(ij)}$, then also $|f_{(U,j)}(x)-f_{(U,j)}(y)|<1$, from which we get $y\in U^j\subset V$.

R e m a r k. The construction shows that the dimension of H depends only upon the cardinality of a base of X or, what is the same, upon the minimal cardinality of a dense subset of X.

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References

[1] Yu. Smirnov, A necessary and sufficient condition for metrizability of a topological space, *Doklady Akad. Nauk SSSR*, **77** (1951), 197—200 (Russian).
[2] R. H. Bing, Metrization of topological spaces, *Canadian J. Math.*, **3** (1951), 175—186.

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