Rings whose proper subrings have property P

By ROBERT GILMER in Tallahassee (Florida, U.S.A.),
ROBERT LEA and MATTHEW O'MALLEY in Houston (Texas, U.S.A.)

Let D be an integral domain with identity. GILMER in [2] determines necessary and sufficient conditions in order that each subring of D with identity be Noetherian. We consider essentially the same type of problem. In general, let R be a ring (not necessarily commutative) and let P be a ring theoretic property. We seek to determine necessary and sufficient conditions on R in order that each proper subring of R has property P.

In this paper we will consider these two properties P:

(P1) Has finite characteristic.

(P2) Has no proper zero divisors.

It is clear that if R satisfies (P1) or (P2), then each proper subring of R satisfies (P1) or (P2), respectively, but neither of the converses is true. Corollary (2. 3) gives a characterization of rings for which each proper subring has finite characteristic, and Corollary (2. 11) characterizes rings for which each proper subring has no proper zero divisors. Moreover, Proposition (2. 5) and Corollary (2. 10) give necessary and sufficient conditions on R in order that each proper two-sided ideal of R satisfies (P1) or (P2), respectively, and Corollaries (2. 7) and (2. 12) give necessary and sufficient conditions on R in order that each proper left (right) ideal of R satisfies (P1) or (P2), respectively.

Section 1 contains the necessary notation and definitions used in the paper and includes the statement of one lemma, which we use frequently. Section 2 contains our main results. Throughout the paper the symbols $\subseteqq$ and $\subset$ will denote containment, and proper containment, respectively. We will use the symbols Z and ω to denote the sets of integers and positive integers, respectively. The authors hereby express their appreciation to CRAIG WOOD for helpful comments concerning this paper. In particular, Wood suggested that the authors work on the problem of finite characteristic.

1. Preliminaries. All rings considered in this paper are assumed to contain more than one element. Throughout the paper, *ideal* will always mean two-sided ideal. If R is a ring and if $\{x_\alpha\}$ is a collection of elements of R, then $[\{x_\alpha\}]$ will denote the subring of R, and $(\{x_\alpha\})$ the ideal of R, generated by $\{x_\alpha\}$. If A is a subring (or ideal) of the ring R, then, following [3; p. 2], we say that A is *genuine* if $A \neq R$, and *proper* if A is genuine and nonzero.

If x is an element of a ring R and if there exists a positive integer n such that $n \cdot x = 0$, then the minimal positive integer for which this is true is called the *order* of x. If no such positive integer exists, we say that x has *infinite order*. If there exists a positive integer n such that $n \cdot x = 0$ for all $x \in R$, the smallest such positive integer is called the *characteristic* of R. If no such positive integer exists, we say that R has *characteristic zero*. If $x \cdot y = 0$ for each $x, y \in R$, we will say that R is the *zero ring* on R^+, the additive group of R; we will also say in this case that R has the *trivial multiplication*.

We will say that an element x of R is a proper *zero divisor* of R if x is nonzero and if there exists a nonzero element a of R such that either xa or ax is zero. A ring R is said to be *simple* if its only (two-sided) ideals are R and (0).

The following lemma is used frequently. It appears as an exercise in [4; p. 101].

(1. 1) Lemma. *If R is a ring with the property that the only left (right) ideals of R are R and* (0), *then R is either:*

(i) *a division ring, or*

(ii) *the zero ring on a finite cyclic group of prime order.*

2. Properties (P1) **and** (P2). Our first concern will be to characterize rings R with the property that each proper subring of R satisfies (P1).

As we have already observed, the condition that R have finite characteristic is not necessary. A zero ring on a quasicyclic group is an example of a ring R with characteristic zero such that each proper subring of R satisfies (P1). We will show that these are the only rings of characteristic zero with the property that each proper subring satisfies (P1).

Before proving the above statement, we give a brief description of a quasicyclic group. (See, for example, [1], [5], [6].) Let be a prime integer. The p-quasicyclic group, which we denote by $C(p^\infty)$, is a group generated by a set $\{c_i\}_{i \in \omega}$ such that c_i has order p^i, and $pc_{i+1} = c_i$, for each $i \in \omega$; there is, to within isomorphism, exactly one group with these properties. (The group of all complex pth power roots of unity, under multiplication, is a realization of $C(p^\infty)$.) The proper subgroups of $C(p^\infty)$ are exactly the finite cyclic groups generated by the c_i's. Thus, with the trivial multiplication, the proper subrings (and proper ideals) of $C(p^\infty)$ are just the proper subgroups of $C(p^\infty)$. It then follows that each proper subring (ideal) of $C(p^\infty)$ has finite characteristic, but $C(p^\infty)$ does not.

(2. 1) Theorem. *Let R be a ring such that every proper subring of R satisfies property* (P1). *R has finite characteristic if any of these five conditions hold:*

(i) *There exists a genuine ideal A of R such that R/A has finite characteristic.*

(ii) *R has an identity element.*

(iii) *R is commutative and contains a maximal ideal.*

(iv) *$R=[r]$ for some $r\in R$.*

(v) *There exist genuine ideals R_1, R_2 of R such that $R=R_1+R_2$.*

Proof. (i) By assumption $nR\subseteq A$ and $mA=(0)$ for some positive integers n and m. Hence $mnR=(0)$, and R has finite characteristic.

(ii) If e is the identity element of R, then e must have finite order, for otherwise $[2e]$ is a proper subring of R, and $[2e]$ does not have finite characteristic.

(iii) Let M be a maximal ideal of R. By Lemma (1. 1), R/M is either a field or R/M is finite. Part (i) of this theorem shows that R has finite characteristic if R/M is a field of finite characteristic or if R/M is finite. If R/M is a field of characteristic zero, choose $r\in R$ such that $r+M$ is the identity element of R/M. Then $nr\notin M$ and $nr^2\notin M$ for any positive integer n; in particular, $nr^2\neq 0$, so that $R=[r]=[r^2]$. Thus, $r=sr^2+mr^2=(sr+mr)r$ for some $s\in R$, $m\in Z$. Since $R=[r]$, it follows easily that $sr+mr$ is an identity element for R, and R has finite characteristic by (ii).

(iv) If $R=[r]$, then R is commutative and any ideal of R maximal with respect to not containing r (such ideals exist by a Zorn's Lemma argument) is a maximal ideal of R. Hence R has finite characteristic by (iii).

(v) If $R=R_1+R_2$, then $n_1R_1=0$ and $n_2R_2=0$ for some positive integers n_1 and n_2. Thus $n_1n_2R=(0)$, and R has finite characteristic.

(2. 2) Theorem. *Suppose that R is a ring such that*

(i) *each genuine ideal of R has finite characteristic,*

(ii) *R does not have finite characteristic, and*

(iii) *each element of R has finite order.*

Then R is the zero ring on a quasicyclic group.

Proof. Let $r\in R-\{0\}$ have order m, and let p be a prime divisor of m. Then $R_p=\{x\in R\mid p^n x=0 \text{ for some } n\in\omega\}$ and $S=\{x\in R\mid \text{the order of } x \text{ is not divisible by } p\}$ are ideals of R, and, as is well known, R is the direct sum of R_p and S. By choice of p, $R_p\neq(0)$; hence part (v) of Theorem (2. 1) implies that $S=(0)$ and $R=R_p$. Moreover, since nR is an ideal of R, it follows from (i) and (ii) that $R=nR$ for each positive integer n. Thus, if $x, y\in R$ and if y has order p^m, then there exists $a\in R$ such that $x=p^m a$. Therefore, $x\cdot y=(p^m a)y=a(p^m y)=0$, so that R has the trivial multiplication.

We show that R is the zero ring on $C(p^\infty)$. Choose $d_1\in R$ such that d_1 has order p. Since $R=pR$, there exists $d_2\in R$ such that $pd_2=d_1$. Then d_2 has order p^2.

Inductively, choose $d_{i+1} \in R$ such that $pd_{i+1} = d_i$ for each $i \in \omega$. Then d_i has order p^i for each $i \in \omega$, and the ideal T generated by $\{d_i\}_{i\in\omega}$ has characteristic zero. Thus, $T=R$. But since R has the trivial multiplication, the ideal T is the same as the additive subgroup generated by $\{d_i\}_{i\in\omega}$. It follows that R is the zero ring on the p-quasicyclic group.

As a corollary to Theorems (2. 1) and (2. 2), we obtain the following result.

(2. 3) Corollary. *Let R be a ring. Each proper subring of R satisfies property* (P1) *if and only if one of the following conditions is satisfied:*

(i) *R has finite characteristic.*

(ii) *R is the zero ring on a quasicyclic group.*

Proof. We have already observed the sufficiency of conditions (i) and (ii). Conversely, if R does not have finite characteristic, then part (iv) of Theorem (2. 1) implies that r has finite order for each $r \in R$. Therefore, by Theorem (2. 2), R is the zero ring on a quasicyclic group.

Since the trivial multiplication defined on any abelian group G induces a ring structure on G, we have the following.

(2. 4) Corollary. (Compare with Ex. 23, p. 22 of [5].) *Let G be an abelian group. If each proper subgroup of G has bounded order, then either G has bounded order or G is a p-quasicyclic group.*

(2. 5) Proposition. *Let R be a ring. Each proper ideal of R satisfies property* (P1) *if and only if one of the following conditions is satisfied:*

(i) *R has finite characteristic.*

(ii) *R is the zero ring on a quasicyclic group.*

(iii) *R is a simple ring having no nonzero element of finite order.*

Proof. The sufficiency is obvious. Suppose that R does not have finite characteristic. If each element of R has finite order, then Theorem (2. 2) implies that R is the zero ring on a quasicyclic group. If there exists an element r of R of infinite order, and if A is the set of elements of R of finite order, then A is a genuine ideal of R, and, by hypothesis, there exists a positive integer k such that $kA=(0)$. However, since nR is an ideal of R, it follows from part (i) of Theorem (2. 1) that $R=nR$ for each positive integer n, and, in particular, $R=kR$. Thus, if $a \in A$, then $a=kr$ for some $r \in R$. But $ka=0$ implies that $k^2r=0$, so that $r \in A$ and $a=kr=0$. Therefore $A=(0)$, so that R is a simple ring with no nonzero element of finite order.

As a consequence of Proposition (2. 5) and its proof, we have the following result.

(2. 6) Corollary. *Let R be a simple ring of characteristic zero. Then R^+, the additive group of R, is isomorphic to a (weak) direct sum of full rational groups.*

Proof. Since R^+ is a divisible, torsion-free abelian group, the result follows from [1; Theorem 19. 1].

(2. 7) Corollary.[1]) *Let R be a ring. Each proper left (right) ideal of R satisfies property* (P1) *if and only if one of the following conditions is satisfied:*

(i) *R has finite characteristic.*

(ii) *R is the zero ring on a quasicyclic group.*

(iii) *R is a division ring of characteristic zero.*

Proof. If R has characteristic zero and if R is not the zero ring on a quasicyclic group, then Proposition (2. 5) implies that R is a simple ring with no nonzero element of finite order. Therefore, R has no proper left or right ideals, so that R is a division ring (Lemma (1. 1)).

We now turn our attention to property (P2). We use the following lemma; its proof is straightforward.

(2. 8) Lemma. *Let R be a ring containing no nonzero nilpotent element. If $a, b \in R$ and if $ab=0$, then $ba=0$, $axb=0$ for each x in R, and $bya=0$ for each y in R. Moreover, $Ra \cap Rb = aR \cap bR = RaR \cap RbR = (0)$.*

(2. 9) Theorem. *Let R be a ring such that R does not satisfy* (P2), *but each proper ideal of R satisfies* (P2).

(i) *If R contains no nonzero nilpotent element, then R is the direct sum of two simple rings, each of which satisfies property* (P2).

(ii) *If R contains a nonzero nilpotent element, then either:*

(a) *R is the zero ring on a cyclic group of prime order, or*

(b) *R is a simple ring such that $R^2=R$.*

Proof. (i) By assumption, there exist nonzero elements a and b of R such that $ab=0$. Since $a^3 \in RaR$ and $b^3 \in RbR$, RaR and RbR are nonzero ideals of R, so that $RaR+RbR = RaR \oplus RbR$ is a nonzero ideal of R that does not satisfy property (P2). Moreover, Lemma (2. 8) implies that RaR and RbR are proper ideals of R, and hence have property (P2). Let S be any nonzero ideal of the ring RaR. Then $RbR \cdot S = S \cdot RbR = (0)$, $S+RbR$ is an ideal of R, and hence $R = S+RbR$. By the modular law, $RaR = RaR \cap (S+RbR) = S+(RaR \cap RbR) = S+(0) = S$. Therefore, (0) and RaR are the only ideals of the ring RaR, so that RaR is a simple ring that satisfies property (P2). Similarly, RbR is a simple ring that satisfies property (P2).

(ii) Suppose that b is a nonzero element of R such that $b^2=0$. Then $(b)=R$. Let M be a genuine ideal of R. If $m \in M$, then bm and mb are elements of M, so that

[1]) The conditions which we obtain here, as well as those of Corollary (2. 12), are reminiscent of the conditions obtained by F. Szász in [7] characterizing rings in which every proper left ideal is cyclic.

$mb \cdot bm = 0$ implies that $bm=0$ or $mb=0$. In particular, $mbm=0$ for each $m \in M$. Thus, $(mb)^2=(bm)^2=0$, and hence $mb=bm=0$ for each $m \in M$. Since $(b)=R$, it follows that for each $r \in R$, $m \in M$, $rm=mr=0$. In particular, $M^2=(0)$, so that $M=(0)$ and R is a simple ring. If $R^2=0$, then R has trivial multiplication, and Lemma (1. 1) implies that R is the zero ring on a cyclic group of prime order. This completes the proof.

As an immediate consequence of Theorem (2. 9), we have the following.

(2. 10) Corollary. *Let R be a ring. Each proper ideal of R satisfies property* (P2) *if and only if one of the following conditions is satisfied:*

(i) *R satisfies property* (P2).

(ii) *R is the direct sum of two simple rings, each of which satisfies property* (P2).

(iii) *R is the zero ring on a cyclic group of prime order.*

(iv) *R does not satisfy* (P2) *and R is a simple ring for which $R^2=R$.*

We now use Theorem (2. 9) to obtain a characterization of rings for which each proper subring satisfies property (P2).

(2. 11) Corollary. *Let R be a ring. Each proper subring of R satisfies property* (P2) *if and only if one of the following conditions is satisfied:*

(i) *R satisfies property* (P2).

(ii) *$R \cong Z/(p) \oplus Z/(q)$, where p and q are prime integers.*

(iii) *R is the zero ring on a cyclic group of prime order.*

Proof. The sufficiency is obvious. Suppose that R does not satisfy property (P2). If x is a nonzero element of R with $x^2=0$, then $(0) \subset [x] = \{\lambda x | x \in Z\}$, and $[x]$ has trivial multiplication. Hence $[x]=R$, and Theorem (2. 9) implies that R is the zero ring on a cyclic group of prime order.

If R contains no nonzero nilpotent element, then Theorem (2. 9) implies that R is the direct sum of two simple rings, R_1 and R_2, each of which satisfies property (P2). Moreover, if S is a nonzero subring of R_1, then $S \cdot R_2 = (0)$, and $R = S + R_2 = R_1 \oplus R_2$. This imples that $S=R_1$, and R_1 is a ring with property (P2) having exactly two subrings R_1 and (0). It follows immediately from Lemma (1. 1) that R_1 is a finite prime field. Similarly, R_2 is a finite prime field, and $R \cong Z/(p) \oplus Z/(q)$, where p and q are prime integers.

(2. 12) Corollary. *Let R be a ring. Each proper left (right) ideal of R satisfies property* (P2) *if and only if one of the following conditions is satisfied:*

(i) *R satisfies property* (P2).

(ii) *R is the zero ring on a cyclic group of prime order.*

(iii) *R is the direct sum of two division rings.*

The proof is similar to that of Corollary (2. 11) and we omit it.

References

[1] L. FUCHS, *Abelian groups* (London, 1960).
[2] R. GILMER, Integral domains with Noetherian subrings, *Comment. Math. Helvetici*, **45** (1970), 129—134.
[3] R. GILMER, *Multiplicative ideal theory* (Kingston, Ontario, 1968).
[4] I. HERSTEIN, *Topics in algebra* (Waltham, 1964).
[5] I. KAPLANSKY, *Infinite abelian groups* (Ann Arbor, 1954).
[6] E. SCHENKMAN, *Group theory* (Princeton, 1965).
[7] F. SZÁSZ, Note on rings in which every proper left ideal is cyclic, *Fundamenta Math.*, **44** (1957), 330—332.

FLORIDA STATE UNIVERSITY
NASA, MANNED SPACECRAFT CENTER

(Received October 6, 1970)