On random multiplicative functions

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1. We call $f(n)$ a completely multiplicative (c.m.) function, if $f(mn)=f(m)f(n)$ holds for all pairs m, n of positive integers. Let $\mathcal{F}$ be the set of those c. m. functions which take the values $+1$ and -1 only.

We say that a function $f(n)\in\mathcal{F}$ is of *normal type*, if

$$\lim_x x^{-1}N\{n\leq x;\ f(n+i)=\varepsilon_i;\ i=0, \dots, k\} = \frac{1}{2^{k+1}} \tag{1.2}$$

for $k=0, 1, 2, \dots$ and for all choices of $\varepsilon_0=\pm 1, \dots;\ \varepsilon_k=\pm 1$.

It would be interesting to give a necessary and sufficient condition for $f(n)$ to be of normal type. Recently E. WIRSING [1] proved that a function $f(n)\in\mathcal{F}$ satisfies (1. 1) with $k=0$ if and only if

$$\sum_{f(p)=-1}\frac{1}{p}=\infty. \tag{1.2}$$

As is easy to see, the validity of (1. 2) is not sufficient for normality. Let for example $f(n)$ be defined as follows: $f(2)=1$, and for an odd prime p let $f(p)=1$ or -1 according as $p\equiv 1$ or $-1 \pmod 4$. Then, by an easy calculation we have

$$\sum_{n\leq x} f(n)f(n+4) = \frac{x}{4}+o(x);$$

hence it follows that $f(n)$ is not a normal function.

We shall see in the following section that almost all multiplicative functions are of normal type. One would think that the Liouville function $\lambda(n)$ is normal. However we can only prove that the system $\lambda(n)=\varepsilon_1$, $\lambda(n+1) = \varepsilon_2$ has infinitely many solutions for an arbitrary choice of $\varepsilon_1=\pm 1$, $\varepsilon_2=\pm 1$. This is a special case of the assertions which we shall prove in the section 3.

2. Let $c, c_1, c_2, \dots$ denote suitable positive constants; let $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$ be arbitrary small positive constants not necessarily the same at every occurrence. Let $d_k(n)$ denote the number of solutions of the equation $n=x_1, \dots, x_k$ in positive integers $x_1, \dots, x_k$, and let $d_2(n)=d(n)$.

Let p_n denote the nth prime number. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\xi_n = \xi_n(\omega)$ $(n=1, 2, \ldots)$ be a sequence of independent random variables with the distribution $P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}$. Let $f(n; \omega)$ be a completely multiplicative function which we define on the set of primes by $f(p_n; \omega) = \xi_n(\omega)$.

We have

Theorem 1. *Almost all $f(n; \omega)$ are of normal type.*

For the proof we need some lemmas.

Lemma 1. *For positive integers C, D let $N(z; D, C)$ denote the number of solutions of the diophantine equation*

$$x^2 - Dy^2 = C \tag{2. 1}$$

in positive integers x, y satisfying $x \leqq z$. Then

$$N(z; D, C) \leqq c_1 d(C^2) \log 2Dz. \tag{2. 2}$$

Perhaps this lemma is known, but I was unable to find a reference to it. We prove now (2. 2).

Without any restriction we can assume that D is a square-free number. For $D=1$ inequality (2. 2) obviously holds, therefore we assume that $D>1$.

Let $K(\sqrt{D})$ denote the quadratic extension field over the rational number-field generated by $\sqrt{D}$. Let R denote the ring of the algebraic integers in $K(\sqrt{D})$, and for a general $\gamma \in R$ let (γ) denote the principal ideal generated by γ.

For a general solution x, y of (2. 1) let $\alpha = x + \sqrt{D}y$, $\beta = x - \sqrt{D}y$. Let $(C) = \pi_1^{\gamma_1} \ldots \pi_r^{\gamma_r}$, where $\pi_1, \ldots, \pi_r$ are different prime ideals. Using the fact that the norm of the ideals is a multiplicative functionand that $N((C)) = C^2$, furthermore that $N(\pi_i)$ is a prime number or a square of a prime number we have $\prod_{i=1}^{r} (\gamma_i + 1) \leqq d(C^2)$. Since $\alpha\beta = C$ and $\alpha, \beta \in R$, therefore $(\alpha)(\beta) = (C)$ and so $(\alpha) | (C)$. Hence it follows that all the solutions can be classified into at most $d(C^2)$ classes, where two solutions x, y; x_1, y_1 belong to the same class if and only if $(\alpha) = (x + \sqrt{D}y) = (\alpha_1) = (x_1 + \sqrt{D}y_1)$. Now we prove that the number of solutions of (2. 1) belonging to a fixed class does not exceed $c_1 \log 2Dz$, whence (2. 2) immediately shall follow.

Let (x_ν, y_ν) $\nu = 0, 1, \ldots, M$ be the all solutions in a class satisfying $1 \leqq x_0 \leqq x_1 \leqq \cdots \leqq x_M \leqq z$, $y_\nu \geqq 0$ and let $\alpha_\nu = x_\nu + y_\nu\sqrt{D}$, $\beta_\nu = x_\nu - y_\nu\sqrt{D}$. We have $(\alpha_0) = (\alpha_1) = \cdots = (\alpha_M)$. Therefore $\alpha_\nu = \alpha_\mu \varepsilon_{\nu\mu}$, $\beta_\nu = \beta_\mu \varrho_{\nu\mu}$, where $\varepsilon_{\nu\mu}$, $\varrho_{\nu\mu}$ are units in R. Since $C = \alpha_\nu \beta_\nu = \alpha_\mu \beta_\mu \varrho_{\nu\mu} \varepsilon_{\nu\mu} = \varrho_{\nu\mu} \varepsilon_{\nu\mu} C$, we have $\varrho_{\nu\mu} = \varepsilon_{\nu\mu}^{-1}$. Using the Dirichlet theorem concerning the form of the units we see that all units have form $\pm\, \varepsilon_0^n$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\varepsilon_0 = \dfrac{u_0 + \sqrt{D}\, v_0}{2}$, and u_0, v_0 are suitable positive integers satisfying $u_0^2 - Dv_0^2 = 4$. Hence $\varepsilon_0 > \frac{\sqrt{D}}{2}$ and we can assume that $\alpha_n = \alpha_0 \varepsilon^n$. Using that $x_n \leqq z$ and that by (2. 1) $y_n \leqq \sqrt{\frac{C+z^2}{D}} \leqq \frac{Cz}{D}$, we have $\alpha_n \leqq (C+1)z$ $(n=0, \dots, M)$. On the other hand, by $\alpha_0 \beta_0 = C$, $0 < \beta_0 < \alpha_0$ we have $\alpha_0 > 1$. Hence $\varepsilon^n < (C+1)z$, whence $M \leqq \frac{\log (C+1)z}{\log \varepsilon_0} \leqq c_1 \log 2Cz$ follows. This completes the proof of Lemma 1.

Corollary. *For positive integers A, B, C let $N(z; A, B, C)$ denote the number of solutions of*

$$Ax^2 - By^2 = C \tag{2. 3}$$

in positive integers x, y, $x \leqq z$. Then

$$N(z; A, B, C) \leqq N(Az; AB, AC) \leqq c_1 d(A^2 C^2) \log 2A^2 Bz.$$

This is obvious. If (x, y) is a solution of (2. 3) then (Ax, y) is a solution of $X^2 - ABY^2 = AC$ which proves the Corollary.

Lemma 2. (Borel—Cantelli) *Let $A_1, A_2, \dots$ be an infinite sequence of sets in (Ω, A, P) and let $\sum_{j=1} P(A_j) < \infty$. Then almost all ω in Ω are belonging to finitely many A_i only.*

Proof of Theorem 1. Let $0 < i_1 < i_2 < \dots < i_k$ be arbitrary but fixed integers. For a general integer n let $\bar{n} = (n+i_1)\dots(n+i_k)$. Let us introduce the notation

$$\eta_N(\omega) = \sum_{n=1}^{N} f(\bar{n}, \omega); \quad M_{l,N} = \int_\Omega (\eta_N(\omega))^l \, dP. \tag{2. 4)-(2. 5}$$

First we give a non-trivial estimation for $M_{4,N}$, whence by using the Borel—Cantelli lemma we deduce that $\lim_{N\to\infty} \eta_N(\omega)/N = 0$ for almost all $\omega \in \Omega$.

It is obvious, that

$$M_{4,N} = \sum_{n_1, n_2, n_3, n_4} \int_\Omega f(\bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4; \omega) \, dP,$$

where in the sum n_1, n_2, n_3, n_4 run independently over the values $1, 2, \dots, N$. Using $\int_\Omega f(m; \omega) dP = 1$ or 0 according to m is a square-number, or not, we have that $M_{4,N}$ is equal to the number of solutions of the equation

$$\bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4 = X^2 \tag{2. 6}$$

in unknowns n_1, n_2, n_3, n_4, X, satisfying $1 \leqq n_i \leqq N$ $(i=1, 2, 3, 4)$.

For a fixed square-free integer $E(>0)$ let $H(E)$ denote the number of solutions of the equation

$$\bar{n}_1\bar{n}_2=EY^2;\quad 1\leqq n_1\leqq N,\quad 1\leqq n_2\leqq N$$

in unknowns n_1, n_2, Y.

It is obvious that if n_1, n_2, n_3, n_4 is a solution of (2. 6) then the square-free parts of the numbers $\bar{n}_1\bar{n}_2$, $\bar{n}_3\bar{n}_4$ are the same. Hence we have

$$M_{4,N}=\sum_E H^2(E),$$

and consequently

$$M_{4,N}\leqq\left(\max_E H(E)\right)\sum_E H(E). \tag{2.7}$$

Observing that $\sum_E H(E)=N^2$ (since the number of the choice of all pairs n_1, n_2, $1\leqq n_i\leqq N$ is N^2) we have

$$M_{4,N}\leqq N^2\max_E H(E). \tag{2.8}$$

Now we estimate $H(E)$. For a general positive square-free A let $G(A)$ denote the number of $n\leqq N$ which can be written in the form

$$\bar{n}=AZ^2, \tag{2.9}$$

where Z is a suitable integer. Then we have

$$H(E)\leqq\sum_{E_1E_2=E}\sum_U G(E_1U)G(E_2U), \tag{2.10}$$

where in the right hand side E_1 runs over the divisors of E and U over the set of all square-free integers coprime to E.

For $k=1$ we evidently have $G(A)\leqq\sqrt{N/A}$. Consequently by (2. 10)

$$H(E)\leqq\sum_{E_1E_2=E}\frac{N}{\sqrt{E}}\cdot\sum_{U\leqq N}\frac{1}{U}\leqq\frac{N\log N}{\sqrt{E}}d(E)\leqq cN\log N,$$

and hence by (2. 8)

$$M_{4,N}\leqq cN^3\log N. \tag{2.11}$$

Assume now that $k\geqq 2$. Consider the solutions of $\bar{n}=AZ^2$. Since the numbers $n+i_{j_1}$, $n+i_{j_2}$ have no common prime-divisors greater than $i_{j_2}-i_{j_1}$ if $j_1\neq j_2$, for an n satisfying (2. 9) we have

$$n+i_j=R_jC_jZ_j^2\qquad (j=1,2,\ldots,k), \tag{2.12}$$

where R_j, C_j are square-free numbers, the prime factors of R_j are not greater than

$i_k - i_1$ and the prime factors of C_j are greater than $i_k - i_1$ and $\prod_{j=1}^{k} C_j | A$. If n is a solution of (2. 12), then

$$i_2 - i_1 = R_2 C_2 Z_2^2 - R_1 C_1 Z_1^2 \tag{2.13}$$

holds with suitable $Z_1, Z_2 \leqq N$. Using the Corollary to Lemma 1 we have that the number of solutions of (2. 13) with $Z_1, Z_2 \leqq N$ is at most $c_1 d((R_1 C_1 (i_2 - i_1))^2) \log N \leqq \leqq c_1 N^{\varepsilon_1}$.

The number of all possible pairs of R_1, R_2 occurring in (2. 12) is bounded for fixed $i_1, i_2, \ldots, i_k$. The number of couples (R_1, R_2) is at most $d^2(A) \leqq c N^{\varepsilon_2}$, since $C_1 C_2 | A$. Therefore

$$G(A) \leqq c N^{\varepsilon}. \tag{2.14}$$

Using (2. 10) and the fact that the number of those A which occur as the square-free part of a number $\bar{n}$ for some $n \leqq N$ is at most N, we have

$$H(E) \leqq c N^{1+\varepsilon}.$$

Hence by (2. 8)

$$M_{4,N} \leqq c N^{3+\varepsilon} \tag{2.15}$$

follows.

Using (2. 11) or (2. 15) according as $k=1$ or $k \geqq 2$, we have

$$P(|\eta_N| > N^{\alpha}) \leqq \int_{\Omega} \frac{|\eta_N|^4}{N^{4\alpha}} \, dP < c N^{3-4\alpha+\varepsilon}. \tag{2.16}$$

Let $N_m = m^5$ and $\alpha = \frac{4}{5} + \varepsilon$. By (2. 16) we have

$$\sum_{m=1}^{\infty} P\left(|\eta_{N_m}| > N_m^{\frac{4}{5}+\varepsilon}\right) \leqq c \sum_{m=1}^{\infty} m^{-1-\varepsilon} < \infty.$$

Consequently by Lemma 2 we have

$$\lim_{m \to \infty} \frac{\eta_{N_m}(\omega)}{N_m^{\frac{4}{5}+2\varepsilon}} = 0 \tag{2.17}$$

for all fixed $\varepsilon > 0$ and for almost all $\omega \in \Omega$. Since for $N_m \leqq N < N_{m+1}$

$$|\eta_N - \eta_{N_m}| \leqq N - N_m \leqq N_{m+1} - N_m \leqq c m^4 \leqq c N_m^{\frac{4}{5}} < c N^{\frac{4}{5}}, \tag{2.18}$$

therefore by (2. 17)

$$\lim_{N \to \infty} \frac{\eta_N(\omega)}{N^{\frac{4}{5}+2\varepsilon}} = 0$$

for all $\varepsilon > 0$ and almost all $\omega \in \Omega$.

Finally we remark, that a function $f(n)\in\mathcal{F}$ is of normal type if and only if $\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{n=1}^{N} f(\bar{n})=0$ for all choice of $k=1, 2, \dots$ and of $(i_1, \dots, i_k)$. This completes the proof of the theorem.

3. Theorem 2. *Let $f(n)$ be a completely multiplicative function, all values of which are $+1$ or -1. Assume that there exist at least two primes p_1, p_2 for which $f(p_1)=f(p_2)=-1$. Then for arbitrary ε_1, ε_2 $(\varepsilon_1=\pm 1,\ \varepsilon_2=\pm 1)$ there exist infinitely many n satisfying $f(n)=\varepsilon_1$, $f(n+1)=\varepsilon_2$.*

For $(\varepsilon_1, \varepsilon_2)=(+1, +1)$ or $(-1, 1)$ we can prove a stronger assertion. This is stated in Theorems 3 and 4.

Theorem 3. *Assuming that the series*

$$\sum_{f(p)=-1}\frac{1}{p} \tag{3.1}$$

diverges we have

$$\liminf_{x\to\infty} x^{-1}N_f(x, 1, 1)\geqq\frac{1}{12}, \tag{3.2}$$

$$\liminf_{x\to\infty} x^{-1}N_f(x, -1, -1)\geqq\frac{1}{12}, \tag{3.3}$$

where $N_f(x, \varepsilon_1, \varepsilon_2)$ denotes the number of those n not exceeding x for which $f(n)=\varepsilon_1$, $f(n+1)=\varepsilon_2$. Consequently

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\leqq x} f(n)f(n+1)\geqq-\frac{5}{6}. \tag{3.4}$$

Let $\mathcal{P}$ be the set of those primes p for which $f(p)=-1$.

Theorem 4. *Suppose that $\mathcal{P}$ contains at least two elements and that the series $\sum\limits_{p\in\mathcal{P}}\frac{1}{p}$ converges. Then for both values $\varepsilon=1, -1$ we have*

$$\lim_{x\to\infty} x^{-1}N_f(x, \varepsilon, \varepsilon)=\frac{1}{4}\left(1+2\varepsilon\prod_{p\in\mathcal{P}}\frac{p-1}{p+1}+\prod_{p\in\mathcal{P}}\frac{p-3}{p+1}\right)\left(\overset{\text{def}}{=}A\right). \tag{3.5}$$

The number standing on the right-hand side of (3.5) *is positive.*

Proof of Theorems 2, 3, and 4. First we prove Theorem 2 for $(\varepsilon_1, \varepsilon_2)=(1, -1)$ and $(-1, 1)$. The remaining two cases will follow from Theorems 3 and 4.

The assertion of Theorem 2 for $(\varepsilon_1, \varepsilon_2)=(1, -1)$ and $(-1, 1)$ is equivalent to saying that $f(n)$ assumes both of the values $+1$ and -1 infinitely many times. For

$+1$ this is true since $f(n^2) = +1$ for all n. To show this for -1 let p be a prime for which $f(p) = -1$. Then $f(p^{2k+1}) = -1$ for all k.

To prove Theorem 3 we need a theorem due to E. WIRSING [1], which we state as

LEMMA 3. *If* $f(n) = \pm 1$ *and the series* (3. 1) *diverges, then*

$$x^{-1} \sum_{x \leqq x} f(n) \to 0 \quad as \quad x \to \infty. \tag{3.6}$$

Let $n_1 < n_2 < \cdots < n_L \leqq x$ be the sequence of those integers for which $f(n_i) = -1$. Let $m_1 < m_2 < \cdots < m_R \leqq x$ denote the complementary sequence, i.e. for which $f(m_i) = +1$. Let $\varrho_k(x)$ denote the number of those n_i for which $n_{i+1} - n_i = k$, $n_i \leqq x$. Similarly, let $\tau_k(x)$ denote the number of m's satisfying $m_{i+1} - m_i = k$, $m_i \leqq x$. From (3. 6) we easily deduce

$$L + O(1) = \sum_{k=1}^{\infty} \varrho_k(x) = \frac{x}{2} + o(x); \quad R + O(1) = \sum_{k=1}^{\infty} \tau_k(x) = \frac{x}{2} + o(x) \tag{3.7}$$

$$\sum_{k=1}^{\infty} k\varrho_k(x) = x + o(x), \quad \sum_{k=1}^{\infty} k\tau_k(x) = x + o(x). \tag{3.8}$$

Hence

$$\sum_{k=3}^{\infty} (k-2)\varrho_k(x) = \varrho_1(x) + o(x), \quad \sum_{k=3}^{\infty} (k-2)\tau_k(x) = \tau_1(x) + o(x) \tag{3.9}$$

follow. Consequently

$$\sum_{k \neq 2} k\varrho_k(x) \leqq 4\varrho_1(x) + o(x), \quad \sum_{k \neq 2} k\tau_k(x) \leqq 4\tau_1(x) + o(x), \tag{3.10}$$

$$\sum_{k \neq 2} \varrho_k(x) \leqq 2\varrho_1(x) + o(x), \quad \sum_{k \neq 2} \tau_k(x) \leqq 2\tau_1(x) + o(x). \tag{3.11}$$

Now we prove that $\liminf_{x \to \infty} x^{-1} \varrho_1(x) \geqq \frac{1}{12}$. The proof of the relation $\liminf_{x \to \infty} x^{-1} \tau_1(x) \geqq \frac{1}{12}$ is similar, and so we omit it. Since from (3. 6)

$$\frac{1}{x} \sum_{2n \leqq x} f(2n) \to 0$$

follows, among the n's the number of even numbers is $\frac{x}{4} + o(x)$. Hence by (3. 11) we have that there are at least $\frac{x}{4} - 2\varrho_1(x) - o(x)$ even n's satisfying $n_{i+1} - n_i = 2$. Let $\mathscr{S}$ denote the set of these n's.

We distinguish two cases.

Case a). $f(2)=1$. Then for $n_i \in \mathcal{S}$ the integers $n_i/2$ and $n_{i+1}/2$ are consecutive numbers, and furthermore $f\left(\frac{n_i}{2}\right) = f\left(\frac{n_{i+1}}{2}\right) = -1$, $\frac{n_i}{2} \leqq \frac{x}{2}$. Thus we have

$$\varrho_1\left(\frac{x}{2}\right) \geqq \frac{x}{4} - 2\varrho_1(x) - o(x),$$

whence $3\varrho_1(x) \geqq \frac{x}{4} - o(x)$, i.e. $\liminf \frac{\varrho_1(x)}{x} \geqq \frac{1}{12}$, follows.

Case b). $f(2) = -1$. Then, for $n_i \in \mathcal{S}$, $\frac{n_i}{2}$ and $\frac{n_{i+1}}{2}$ are consecutive integers, and moreover $f\left(\frac{n_i}{2}\right) = f\left(\frac{n_{i+1}}{2}\right) = +1$, $\frac{n_i}{2} \leqq \frac{x}{2}$. Consequently

$$\tau_1\left(\frac{x}{2}\right) \geqq \frac{x}{4} - 2\varrho_1(x) + o(x). \tag{3.12}$$

Since the interval $[m_i, m_{i+1}]$ for $m_{i+1} - m_i = k$, $k \geqq 3$ contains $(k-1)$ elements from among the n's, we deduce that

$$\varrho_1(x) \geqq \sum_{k=3}^{\infty} (k-2)\tau_k(x);$$

hence by (3. 12)

$$\varrho_1(x) \geqq \tau_1(x) + o(x) \tag{3.13}$$

follows. From here by (3. 12) we obtain

$$3\varrho_1(x) \geqq \frac{x}{4} + o(x),$$

i.e.

$$\lim_{x\to\infty} x^{-1}\varrho_1(x) \geqq \frac{1}{12}.$$

Now we prove Theorem 4. For this we need the following

Lemma 4. [2] *If $h(n)$ is a complex-valued completely multiplicative function satisfying the conditions: a) $|h(n)| \leqq 1$ $(n = 1, 2, \dots)$, and b) $\sum_p \frac{h(p)-1}{p}$ converges, then*

$$\lim_{x\to\infty} x^{-1} \sum_{n\leqq x} h(n)h(n+1) = \prod_p \left(1 + 2\sum_{\alpha=1}^{\infty} \frac{h(p^\alpha) - h(p^{\alpha-1})}{p^\alpha}\right),$$

$$\lim_{x\to\infty} x^{-1} \sum_{n\leqq x} h(n) = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{h(p^\alpha) - h(p^{\alpha-1})}{p^\alpha}\right).$$

Observing that the conditions of Lemma 4 are satisfied for $h(n)=f(n)$ and that

$$4N(x,\varepsilon,\varepsilon)=\sum_{n\leqq x}(f(n)+\varepsilon)(f(n+1)+\varepsilon)=\sum_{n\leqq x}f(n)f(n+1)+2\varepsilon\sum_{n\leqq x}f(n)+x+0(1),$$

by Lemma 4 we obtain (3. 5).

Finally we prove the positivity of A. If $3\in\mathcal{P}$, then

$$A\geqq\frac{1}{4}\left(1-2\cdot\frac{2}{4}\prod_{p\in\mathcal{P},\,p\neq 3}\frac{p-1}{p+1}\right).$$

Since $\prod_{\substack{p\in\mathcal{P}\\ p\neq 3}}$ is not an empty product, it must be smaller than 1; so indeed $A>0$. If $3\notin P$, $2\in\mathcal{P}$, then

$$A\geqq\frac{1}{4}\left(1-\frac{2}{3}\prod_{p\in\mathcal{P},\,p>3}\frac{p-1}{p+1}-\frac{1}{3}\prod_{p\in\mathcal{P},\,p>3}\frac{p-3}{p+1}\right).$$

Using the fact that the products on the right hand side are not empty, we again have $A>0$. If 2, 3 are not belonging to $\mathcal{P}$, then

$$A\geqq\frac{1}{4}\left(1-2\prod_{p\in\mathcal{P},\,p>3}\frac{p-1}{p+1}+\prod_{p>3}\frac{p-3}{p+1}\right).$$

Using the relation $\frac{p-3}{p+1}<\left(\frac{p-1}{p+1}\right)^2$ for $p\geqq 3$,

$$A\geqq\frac{1}{4}\left(1-\prod_{p\in\mathcal{P}}\frac{p-1}{p+1}\right)^2>0$$

also in this case.

References

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