Products of contractions in Hilbert space

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The aim of this Note is to study the convergence of some infinite products of contractions acting on a Hilbert space. We extend the results of I. HALPERIN ([4], Th. 1) and F. BROWDER ([1], Lemma 3) concerning products of projections to a larger class of operators.

Throughout this Note, H will denote a Hilbert space, $\mathcal{S}(H)$ the unit ball of the algebra of all bounded linear operators acting on H, and I the identity operator on H. The set of all natural numbers is denoted by N and if $T_j \in \mathcal{S}(H)$ $(j \in N)$ we set $T_n T_{n-1} \cdots T_1 = \prod_{j=1}^{n} T_j$. We also set for $\varepsilon \geqq 0$ and $T \in \mathcal{S}(H)$:

$$\varphi_T(\varepsilon) = \sup_{\substack{\|x\| \leqq 1 \\ \|x\| - \|Tx\| \leqq \varepsilon}} \|x - Tx\|, \quad \text{and} \quad \mathcal{S}_\varphi(H) = \{T \in \mathcal{S}(H) : \lim_{\varepsilon \to 0} \varphi_T(\varepsilon) = 0\}.$$

For any $T \in \mathcal{S}(H)$ the function φ_T is increasing and if $\|x\| \leqq 1$ we have

$$\|x - Tx\| \leqq \sup \{\|y - Ty\| : \|y\| \leqq 1, \ \|y\| - \|Ty\| \leqq \|x\| - \|Tx\|\} = \varphi_T(\|x\| - \|Tx\|).$$

Definition. A map ψ defined on N and valued in an arbitrary set is called *permissible* if for any $k \in N$ there is an $r_k \in N$ such that $\psi(k)$ belongs to the image by ψ of each block of r_k successive natural numbers (see [1], Def. 6).

Lemma 1. *Let* $T, S \in \mathcal{S}_\varphi(H)$. *We have* $TS \in \mathcal{S}_\varphi(H)$, $\operatorname{Ker}(I - TS) = \operatorname{Ker}(I - T) \cap \operatorname{Ker}(I - S)$.

Proof. For any $x \in H$ such that $\|x\| \leqq 1$, $\|x - TSx\| \leqq \varepsilon$ we have

$$\|x - TSx\| \leqq \|x - Tx\| + \|Tx - TSx\| \leqq \|x - Tx\| + \|x - Sx\| \leqq \varphi_T(\|x\| - \|Tx\|) + \\ + \varphi_S(\|x\| - \|Sx\|).$$

Using the inequalities $\|TSx\| \leqq \|Sx\|$, $\|TSx\| \leqq \|T(Sx - x)\| + \|Tx\|$ we also obtain

$$\|x\| - \|Sx\| \leqq \|x\| - \|TSx\| \leqq \varepsilon,$$

$$\|x\| - \|Tx\| \leqq \|T(x - Sx)\| + \|x\| - \|TSx\| \leqq \varphi_S(\varepsilon) + \varepsilon.$$

It results $\varphi_{TS}(\varepsilon) \leqq \varphi_T(\varphi_S(\varepsilon)+\varepsilon)+\varphi_S(\varepsilon)$, which implies $\lim \varphi_{TS}(\varepsilon)=0$ as $\varepsilon \to 0$, and $TS \in \mathcal{S}_\varphi(H)$.

Now if $x=TSx$, $\|x\| \leqq 1$ we have $\|x\|=\|Sx\|$, $\|x-Sx\| = \lim_{\varepsilon\to 0} \varphi_S(\varepsilon) = 0$, thus $x=Sx=Tx$, $\operatorname{Ker}(I-TS) \subset \operatorname{Ker}(I-T) \cap \operatorname{Ker}(I-S)$. The opposite inclusion is evident.

Lemma 2. *Let $\mathcal{A} \subset \mathcal{S}_\varphi(H)$ such that $\lim_{\varepsilon\to 0} \sup_{T\in\mathcal{A}} \varphi_T(\varepsilon)=0$. Then for any sequence $\{T_j\}_{j\in N}$ we have $\lim_{n\to\infty} \|(\prod_{j=1}^{n} T_j - \prod_{j=1}^{n+1} T_j)x\| = 0$.*

Proof. Let $x \in H$, $\|x\| \leqq 1$. Since $\{\|\prod_{j=1}^{n} T_j x\|\}_{n\in N}$ is a convergent sequence one obtains

$$\lim_{n\to\infty} \|(\prod_{j=1}^{n} T_j - \prod_{j=1}^{n+1} T_j)x\| \leqq \lim_{n\to\infty} \varphi_T(\|\prod_{j=1}^{n} T_j x\| - \|\prod_{j=1}^{n+1} T_j x\|) \leqq$$

$$\leqq \lim_{n\to\infty} \sup_{T\in\mathcal{A}} \varphi_T(\|\prod_{j=1}^{n} T_j x\| - \|\prod_{j=1}^{n+1} T_j x\|) = 0.$$

Theorem 1. *Let $\mathcal{A}$ be a commutative subset of $\mathcal{S}_\varphi(H)$ and $\psi: N \to \mathcal{A}$ such that $\psi^{-1}(\psi(k))$ is an infinite set for any $k \in N$. Then $\{\prod_{j=1}^{n} T_j\}_{n\in N}$ converges strongly to the orthogonal projection P of H onto $\bigcap_{k=1}^{\infty} \operatorname{Ker}(I-\psi(k))$.*

Proof. For simplicity we introduce the notation $T_n = \prod_{j=1}^{n} \psi(j)$, $N_k=\psi^{-1}(\psi(k))$. We have $\langle T_{n+1}^* T_{n+1} x, x\rangle = \langle T_n^* \psi(n+1)^* \psi(n+1) T_n x, x\rangle \leqq \langle T_n^* T_n x, x\rangle$, thus by [6], Sec. 104 (p. 261) the sequence $\{T_n^* T_n\}_{n\in N}$ converges strongly to a positive operator A.

Let $y \in H$, $\|y\| \leqq 1$. We have

$$\lim_{n+1\in N_k} \|(I-\psi(k)T_n y\| \leqq \lim_{n+1\in N_k} \varphi_{\psi(k)}(\|T_n y\| - \|\psi(k) T_n y\|) =$$

$$= \lim_{n+1\in N_k} \varphi_{\psi(k)}(\|T_n y\| - \|T_{n+1} y\|) = 0,$$

consequently

$$\langle (I-\psi(k)^*)Ax, y\rangle = \lim_{n+1\in N_k} \langle (I-\psi(k)^*) T_n^* T_n x, y\rangle =$$

$$= \lim_{n+1\in N_k} \langle T_n x, (I-\psi(k)) T_n y\rangle = 0.$$

Because the kernels of $I-\psi(k)$ and $I-\psi(k)^*$ coincide (see [5], Sec. I. 3. 1) we infer

$$(I-\psi(k)A = (I-\psi(k)^*)A = 0, \ T_n^* T_n A = A$$

which shows that A is a projection and $A \leqq P$. But we have also $\langle Px, x\rangle = \langle T_n^* P T_n x, x\rangle \leqq \langle T_n^* T_n x, x\rangle$ thus $P \leqq A$. It follows $P = A$ and

$$\lim_{n\to\infty} \|T_n x - Px\|^2 = \lim_{n\to\infty} \langle T_n^* T_n x - Px, x\rangle = \langle Ax - Px, x\rangle = 0$$

which concludes the proof.

Corollary 1. *Let* $T_j \in \mathcal{S}_\varphi(H)$, $j = 1, 2, \ldots, m$ *and put* $T = T_1, T_2, \ldots, T_m$. *Then* $\{T^n\}_{n\in N}$ *converges strongly to the orthogonal projection of* H *onto* $\bigcap_{j=1}^{m} \operatorname{Ker}(I - T_j)$.

Proof. By lemma 1 we have $T \in \mathcal{S}_\varphi(H)$, $\operatorname{Ker}(I-T) = \bigcap_{j=1}^{m} \operatorname{Ker}(I - T_j)$. Set $\mathcal{A} = \{T\}$, $\psi(j) = T$. We have $T^n = \prod_{j=1}^{n} \psi(j)$ thus we can apply Th. 1.

Theorem 2. *Let* $\mathcal{A} \in \mathcal{S}_\varphi(H)$ *such that* $\lim_{\varepsilon\to 0} \sup_{T\in\mathcal{A}} \varphi_T(\varepsilon) = 0$ *and let* $\psi : N \to \mathcal{A}$ *be a permissible map. Then* $\{\prod_{j=1}^{n} \psi(j)\}_{n\in N}$ *converges weakly to the orthogonal projection of* H *onto* $\bigcap_{j=1}^{\infty} \operatorname{Ker}(I - \psi(j))$.

Proof. Let P be the orthogonal projection of H onto $\bigcap_{j=1}^{\infty} \operatorname{Ker}(I - \psi(j))$. We have $\psi(j)P = P$, $\psi(j)^* P = P$, thus $\prod_{j=1}^{n} \psi(j) P = P \prod_{j=1}^{n} \psi(j) = P$ and

$$\lim_{n\to\infty} \langle (\prod_{j=1}^{n} \psi(j) - P)x, y\rangle = \lim_{n\to\infty} \langle (I-P) \prod_{j=1}^{n} \psi(j) x, y\rangle .$$

Suppose the subsequence $\{\prod_{j=1}^{m_n} \psi(j) x\}_{n\in N}$ converges weakly to z and take $k \in N$. By the definition of ψ there is $r_k \in N$ and $s_n \in N$ such that $\psi(s_n) = \psi(k)$, $r_k \geqq m_n - s_n \geqq 0$. Using Lemma 2 we get

$$z = \lim_{n\to\infty} \prod_{j=1}^{m_n} \psi(j) x = \lim_{n\to\infty} \prod_{j=1}^{s_n - 1} \psi(j) x = \lim_{n\to\infty} \prod_{j=1}^{s} \psi(j) x = \psi(k) z;$$

consequently $Pz = z$.

Now if $\{\prod_{j=1}^{n} \psi(j)\}$ does not converge weakly to P we can find $x, y \in H$ and a subsequence $\{m_n\}_{n\in N}$ such that $\lim_{n\to\infty} \langle (\prod_{j=1}^{m_n} \psi(j) - P)x, y\rangle = a \neq 0$.

But we may suppose that $\{\prod_{j=1}^{m_n} \psi(j) x\}_{n\in N}$ converges weakly to $z = Pz$ because balls in H are weakly compact and we get a contradiction:

$$0 \neq a = \lim_{n\to\infty} \langle (I-P) \prod_{j=1}^{m_n} \psi(j) x, y\rangle = \langle (I-P)z, y\rangle = 0.$$

Remark. The set of all orthogonal projections in H is contained in $\mathcal{S}_\varphi(H)$. If $P(=P^*)$ is a projection we have $\varphi_P(\varepsilon)\leqq\sqrt{2\varepsilon}$ thus Cor. 1 and Th. 1. are equally applicable to projection operators (results of I. HALPERIN and F. BROWDER). Let σ be a closed set in the complex plane included in the unit disc D and such that $\sigma\cap\{\lambda:|\lambda|=1\}$ contains at most the point 1. If T is a normal operator, $\sigma(T)\subset\sigma$, with the spectral measure $E(\cdot)$, let us put $\sigma_\varepsilon=\left\{\lambda\in\sigma:|\lambda|\leqq\sqrt{1-\sqrt{\varepsilon}}\right\}$, $E(\sigma_\varepsilon)=P_\varepsilon$, $I-E(\sigma_\varepsilon)=Q_\varepsilon$. For any $x\in H$, $\|x\|\leqq 1$, $\|x\|-\|Tx\|\leqq\varepsilon$ we have

$$2\varepsilon\geqq\|x\|^2-\|Tx\|^2=\|P_\varepsilon x\|^2-\|TP_\varepsilon x\|^2+\|Q_\varepsilon x\|^2-\|TQ_\varepsilon x\|^2\geqq\|P_\varepsilon x\|^2-$$
$$-\|TP_\varepsilon x\|^2\geqq\sqrt{\varepsilon}\,\|P_\varepsilon x\|^2$$

thus $\|P_\varepsilon x\|\leqq\sqrt{2\sqrt{\varepsilon}}$ and $\|x-Tx\|\leqq 2\sqrt{2\sqrt{\varepsilon}}+\|Q_\varepsilon x-TQ_\varepsilon x\|\leqq 2\sqrt{2\sqrt{\varepsilon}}+\|(T|Q_\varepsilon H)\|$. It follows $\varphi_T(\varepsilon)\leqq\sqrt{2\sqrt{\varepsilon}}+\sup_{\lambda\in\sigma-\sigma_\varepsilon}|1-\lambda|$ thus $T\in\mathcal{S}_\varphi(H)$. In fact we showed that Th. 2 is applicable to the set $\mathcal{A}=\{T\in\mathcal{S}(H);\ T \text{ normal},\ \sigma(T)\subset\sigma\}$.

References

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