On a certain class of representations of function algebras

By DUMITRU GAŞPAR in Timişoara (S. R. Romania)

1. Introduction. In [4] B. SZ.-NAGY and C. FOIAŞ have introduced the class $\mathscr{C}_\varrho$ of all bounded linear operators T on the Hilbert space H, which admit a representation of the form:

$$T^n = \varrho P_H U^n | H \qquad (n = 1, 2, \ldots), \tag{1}$$

where U is a unitary operator on a Hilbert space K, containing H as a subspace and P_H is the orthogonal projection of K onto H. In [3] they have proved that any T belonging to some class $\mathscr{C}_\varrho$ is similar to a contraction.

The definition of the class $\mathscr{C}_\varrho$ has a natural correspondent for operator valued representations on Hilbert spaces. For this let X be a compact Hausdorff space, $C(X)$ the Banach algebra of all complex-valued continuous functions on X, A a function algebra on X (i. e. a closed subalgebra of X, which contains the constants and separates the points of X), and M_A the maximal ideal space of A (i.e. the set of all complex homomorphisms of A). For any $\Phi \in M_A$ there exists a positive measure m on X such that

$$\Phi(f) = \int f \, dm \qquad (f \in A).$$

Such a measure is called a representing measure for Φ (see [6]). As usual we write A_Φ for the kernel of Φ.

By a *representation* of A on H we shall mean an algebraic homomorphism $f \to T_f$ of A in $\mathscr{B}(H)$ (the algebra of all bounded linear operators on H) satisfying $T_1 = I$ (the identical operator on H) and

$$\|T_f\| \leq k \|f\| \qquad (f \in A).$$

If $k = 1$, $f \to T_f$ is called a *contractive representation* of A on H.

Let $\varrho > 0$. A (contractive) representation $\varphi \to U_\varphi$ of $C(X)$ on a Hilbert space K, where K contains H as a subspace, *will be called a spectral ϱ-dilation of $f \to T_f$ with respect to $\Phi \in M_A$, if*

$$T_f = \varrho P_H U_f | H \qquad (f \in A_\Phi). \tag{2}$$

We say that a representation of A on H is of *class* $\mathscr{C}_\varrho(A, H)$ if it has a spectral ϱ-dilation. If $\varrho=1$, the spectral ϱ-dilation of $f\to T_f$ means simply the *spectral dilation* of $f\to T_f$ (see [2]). A contractive representation for which there exists a spectral dilation is called a *dilatable representation.*

The purpose of this note is to prove the analog of the result in [3], in the context of representations of function algebras. This is contained in the following

Theorem. *Let $f\to T_f$ be a representation of class $\mathscr{C}_\varrho(A, H)$ with respect to $\Phi\in M_A$. Then there exists a Hilbert space H', an affinity X of H' onto H, and a contractive representation $f\to T'_f$ of A on H' such that*

$$T_f X = X T'_f \qquad (f\in A).$$

Moreover, $f\to T'_f$ is a dilatable representation, and the spectral ϱ-dilation of $f\to T_f$ is a spectral dilation of $f\to T'_f$.

2. Firstly we get a caracterization of the classes $\mathscr{C}_\varrho(A, H)$ and the monotonity of these classes. For this aim let $f\to T_f$ be a representation of class $\mathscr{C}_\varrho(A, H)$ and $\varphi\to U_\varphi$ its spectral ϱ-dilation. If $f\in A$, relation (2) implies:

$$\varrho P_H U_f|H = \varrho P_H U_{f-\Phi(f)}|H + \varrho\Phi(f) I = T_f + (\varrho-1)\Phi(f) I,$$

that is,

(3)
$$\frac{1}{\varrho} T_f + \left(1-\frac{1}{\varrho}\right)\Phi(f) I = P_H U_f|H \qquad (f\in A).$$

Now $\varphi\to S_\varphi = P_H U_\varphi|H$ $(\varphi\in C(X))$ is a positive map of $C(X)$ into $\mathscr{B}(H)$ (see [1]) for which the spectral dilation is exactly $\varphi\to U_\varphi$. Now T_f has the form:

$$T_f = \varrho S_f + (1-\varrho)\Phi(f) I = \varrho S_f + (1-\varrho)\left(\int f\, dm\right) I,$$

where m is a fixed representing measure for Φ.

If we put

$$\tilde{T}_\varphi = \varrho S_\varphi + (1-\varrho)\left(\int \varphi\, dm\right) I \qquad (\varphi\in C(X))$$

we obtain a linear map $\varphi\to\tilde{T}_\varphi$ of $C(X)$ into $\mathscr{B}(H)$, which extends the given representation and satisfies

$$\frac{1}{\varrho}\tilde{T}_\varphi + \left(1-\frac{1}{\varrho}\right)\left(\int\varphi\, dm\right) I \geqq 0 \qquad (\varphi\geqq 0,\ \varphi\in C(X)).$$

The last condition is equivalent to

(4)
$$(\varrho-1)\left(\int\varphi\, dm\right) I + \tilde{T}_\varphi \geqq 0 \qquad (\varphi\geqq 0,\ \varphi\in C(X)).$$

Conversely if we are given a representation $f\to T_f$ of A on H, which admits an extension $\varphi\to\tilde{T}_\varphi$ to $C(X)$ satisfying (4), then

$$S_\varphi=\frac{1}{\varrho}\tilde{T}_\varphi+\left(1-\frac{1}{\varrho}\right)\left(\int\varphi\,dm\right)I$$

defines a positive map $\varphi\to S_\varphi$ of $C(X)$ into $\mathscr{B}(H)$. Let $\varphi\to U_\varphi$ be the spectral dilation of $\varphi\to S_\varphi$ (see [1]). It is immediate that $\varphi\to U_\varphi$ is a spectral ϱ-dilation of $f\to T_f$, and consequently the given representation is of class $\mathscr{C}_\varrho(A, H)$. In this manner we have proved the following

Proposition. *The representation $f\to T_f$ of A on H is of the class $\mathscr{C}_\varrho(A, H)$ if and only if it admits a linear extension $\varphi\to\tilde{T}_\varphi$ to $C(X)$ satisfying* (4).

Corollary. *If $\varrho\leqq\varrho'$ then $\mathscr{C}_\varrho(A, H)\subseteqq\mathscr{C}_{\varrho'}(A, H)$.*

Proof. Let $f\to T_f$ be a representation of the class $\mathscr{C}_\varrho(A, H)$. Then, by the proposition, it has an extension $\varphi\to\tilde{T}_\varphi$ to $C(X)$ which satisfies (4). But if $\varphi\in C(X)$, $\varphi\geqq 0$, then for $\varrho'\geqq\varrho$ we have $(\varrho'-1)\left(\int\varphi\,dm\right)I+\tilde{T}_\varphi\geqq(\varrho-1)\left(\int\varphi\,dm\right)I+\tilde{T}_\varphi\geqq 0$, that is, condition (4) is satisfied, with ϱ' instead of ϱ. According to the above proposition, $f\to T_f$ is of the class $\mathscr{C}_{\varrho'}(A, H)$, and the corollary is proved.

3. Now we are able to prove the theorem. This proof is modelled on that in [3]. In the sequel m will be a fixed representing measure for Φ.

We suppose that $f\to T_f$ is of class $\mathscr{C}_r(A, H)$. Then, by the corollary, it is also of class $\mathscr{C}_\varrho(A, H)$ for $\varrho\geqq r$. Let $\varphi\to U_\varphi$ be the spectral ϱ-dilation of $f\to T_f$, and K_ϱ the ϱ-dilation space. We set

$$M_\varrho=\bigvee_{f\in A_\Phi,\, g\in A} U_g^*(U_f^*-T_f^*)H \tag{5}$$

and $t_\varrho=\|P_{M_\varrho}|H\|$, where P_{M_ϱ} is the orthogonal projection of K_ϱ on M_ϱ. It is obvious that $t_\varrho\leqq 1$. Moreover, t_ϱ is the smallest positive number for which the inequality

$$|(h, m_\varrho)|\leqq t_\varrho\|h\|\,\|m_\varrho\| \tag{6}$$

holds for any $h\in H$ and $m_\varrho\in M_\varrho$ of the form:

$$m_\varrho=\sum_{g,f} U_g^*(U_f^*-T_f^*)h_g^f, \tag{7}$$

where the family $\{h_g^f: g\in A,\ f\in A_\Phi\}$ has a finite number of elements.

Using (3) we obtain by a simple computation:

$$(h, m_\varrho)=\left(h, \sum_{g,f}(\delta-1)\overline{\Phi(g)}T_f^*h_g^f\right),$$

where $\delta=\frac{1}{\varrho}$. Consequently, relation (6) is equivalent to

(8) $$(\delta-1)^2\left\|\sum_{g,f}\overline{\Phi(g)}T_f^* h_g^f\right\|^2 \leqq t_\varrho^2\|m_\varrho\|^2.$$

Now we compute the norm of m_ϱ:

$$\|m_\varrho\|^2=\sum_{g,g'}\Big(U_{g'\bar g}\sum_f(U_f^*-T_f^*)h_g^f,\ \sum_{f'}(U_{f'}^*-T_{f'}^*)h_{g'}^{f'}\Big)=$$

$$=\sum_{g,g'}\Big[\sum_{f,f'}(U_{f'g'\bar g\bar f}h_g^f, h_{g'}^{f'})-\sum_{f,f'}(T_f^*h_g^f, U_{g\bar g'\bar f'}h_{g'}^{f'})-$$

$$-\sum_{f,f'}(U_{g'\bar g\bar f}h_g^f, T_{f'}^*h_{g'}^{f'})+\sum_{f,f'}(U_{g'\bar g}T_f^*h_g^f, T_{f'}^*h_{g'}^{f'})\Big]=$$

$$=\sum_{\substack{g,g'\\f,f'}}(h_g^f, h_{g'}^{f'})\int f'g'\bar f\bar g\,dm-2\operatorname{Re}\sum_{\substack{g,g'\\f,f'}}(T_f^*h_g^f, h_{g'}^{f'})\int f'g'\bar g\,dm+$$

$$+\sum_{g,g'}(T_f^*h_g^f, T_{f'}^*h_{g'}^{f'})\int g'\bar g\,dm+\frac{1}{\varrho}\sum.$$

In this calculus we have used:

$$(U_\varphi h, h')=(h,h')\int\varphi\,dm+\frac{1}{\varrho}\Big[(\tilde T_\varphi h, h')-(h,h')\int\varphi\,dm\Big]\qquad (h,h'\in H;\ \varphi\in C(X))$$

and we have denoted by $\frac{1}{\varrho}\Sigma$ the term which contains $\frac{1}{\varrho}$ as a factor.

By introducing the scalar products under the integral and interchanging the sum with the integral it follows

$$\|m_\varrho\|^2=\int\Big\{\Big\|\sum_{g,f}\bar f\bar g h_g^f\Big\|^2-2\operatorname{Re}\Big(\sum_{g,f}\bar g T_f^*h_g^f,\ \sum_{g',f'}\bar g'f'h_{g'}^{f'}\Big)+\Big\|\sum_{g,f}\bar g T_f^*h_g^f\Big\|^2\Big\}dm+\frac{1}{\varrho}\sum=$$

$$\int\Big\|\sum_{g,f}\bar f\bar g h_g^f-\sum_{g,f}\bar g T_f^*h_h^f\Big\|^2dm+\frac{1}{\varrho}\sum.$$

Now writing $m_r\in M_r$ as in (7) we obtain

(9) $$\varrho\|m_\varrho\|^2-r\|m_r\|^2=(\varrho-r)\int\Big\|\sum_{g,f}\bar f\bar g h_g^f-\sum_{g,f}\bar g T_f^*h_g^f\Big\|^2dm.$$

By (9) and by a simple evaluation of the integral of the vector-valued continuous functions we deduce

$$\varrho\|m_\varrho\|^2\geqq r\|m_r\|^2+(\varrho-r)\Big\|\int\sum_{g,f}(\bar f\bar g h_g^f-\bar g T_f^*h_g^f)\,dm\Big\|^2=$$

$$=r\|m_r\|^2+(\varrho-r)\Big\|\sum_{g,f}\Big(\int\bar g\,dm\Big)T_f^*h_g^f\Big\|^2.$$

For the last equality we have used

$$\int \bar{f}\bar{g}h_g^f\,dm = \left(\int \bar{f}\bar{g}\,dm\right)h_g^f = \overline{\Phi(f)}\,\overline{\Phi(g)}h_g^f = 0.$$

Because (8) remains true if $\varrho=r$, with 1 instead of t_r we have

$$\varrho\|m_\varrho\|^2 \geqq \left[r\left(\frac{1}{r}-1\right)^2+(\varrho-r)\right]\|\sum_{g,f}\overline{\Phi(g)}T_f^*h_g^f\|^2 =$$

$$= \left(\varrho-2+\frac{1}{r}\right)\|\sum_{g,f}\overline{\Phi(g)}T_f^*h_g^f\|^2.$$

Now by multiplying with $\left(\frac{1}{\varrho}-1\right)^2$, a simple computation shows that

$$\left(1-\frac{1}{\varrho}\right)^2\|\sum_{g,f}\overline{\Phi(g)}T_f^*h_g^f\|^2 \leqq \frac{\varrho-2+\frac{1}{\varrho}}{\varrho-2+\frac{1}{r}}\|m_\varrho\|^2.$$

Comparing this inequality with (8) we conclude that $t_\varrho<1$ for $\varrho>r$.

The rest of the proof proceeds exactly the same way as in [3], with the only remark that $k\in N_\varrho=K_\varrho\ominus M_\varrho$ $(\varrho>r)$ if and only if

$$T_f P_H U_g k = P_H U_{gf} k \qquad (g\in A,\ f\in A_\Phi).$$

The desired space in the theorem is $H'=P_{N_\varrho}H$, the affinity is $X=\overline{P_H|H'}$, and finally $T_f'=P_{H'}U_f|H'$ $(f\in A)$.

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