

A theorem on boundary spectra

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Dedicated in sorrow to the memory of David Topping

1. Let T be a bounded linear operator on a Hilbert space H . For a complex number ζ , let $E_\zeta[T] = \{x \in H: Tx = \zeta x\}$; thus $E_\zeta[T] \neq (0)$ if and only if ζ is an eigenvalue of T . As well known, the relation

$$(1) \quad E_\zeta[T] = E_{\bar{\zeta}}[T^*]$$

holds for normal operators, but does not hold in general for non-normal operators; in fact, it is obvious that no point ζ of the residual spectrum of T can satisfy (1). It does hold if $\|T\| \leq 1$ and $|\zeta| = 1$, see [6] or [5; p. 8]. SCHREIBER [7] generalized this result as follows. Let $A_\zeta[T] = \{x_n: x_n \in H, \|x_n\| = 1, \|Tx_n - \zeta x_n\| \rightarrow 0 \ (n \rightarrow \infty)\}$. Then

$$(2) \quad A_\zeta[T] = A_{\bar{\zeta}}[T^*]$$

holds if $\|T\| \leq 1$ and $|\zeta| = 1$.

SZ.-NAGY and C. FOIAŞ [4] proved the following theorem.

Theorem A. *Let X be a spectral set for T , and let ζ be a boundary point of X . If there exists a sequence $\{D_n\}$ of open disks $D_n = \{\alpha: |\alpha - \alpha_n| < r_n\}$ contained in the complement of X such that*

$$\alpha_n \rightarrow \zeta, \quad r_n^{-1} |\alpha_n - \zeta| \rightarrow 1 \quad \text{as } n \rightarrow \infty,$$

then (1) is true for T and ζ .

If T is a contraction, the unit disk $X = \{\lambda: |\lambda| \leq 1\}$ is a spectral set for T , and this theorem is a generalization of the above result for contractions. In the light of this fact it is natural to seek a generalization of (2).

2. In the following, we refer to the condition on ζ in Theorem A as condition A_X . Then our result is the following:

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Theorem 1. Let X be a closed set containing the spectrum $\sigma(T)$ of an operator T , and suppose that

$$(3) \quad \|(T - \alpha I)^{-1}\| \leq 1/d(\alpha, X)$$

for $\alpha \notin X$, where $d(\alpha, X)$ is the distance from α to X . Then relation (2) (and hence (1)) holds for any point $\zeta \in X$ satisfying condition A_X .

If X is a spectral set for T , i.e. if

$$\|f(T)\| \leq \sup \{|f(\lambda)| : \lambda \in X\}$$

for all rational functions $f(\lambda)$ without poles in X , then T obviously satisfies (3).

If T is a hyponormal operator, condition (3) is satisfied. For, in this case, $(T - \alpha I)^{-1}$ ($\alpha \notin X$) is also hyponormal [8], and so $(T - \alpha I)^{-1}$ is normaloid for $\alpha \notin X$. Thus we have

$$\|(T - \alpha I)^{-1}\| = 1/\inf \{|\lambda - \alpha| : \lambda \in \sigma(T)\} \leq 1/\inf \{|\lambda - \alpha| : \lambda \in X\} = 1/d(\alpha, X).$$

A recent result [3] of PUTNAM is essentially the same as Theorem A, and Theorem 1 is also a generalization of [3] to the case of approximate eigenvalues.

The following proof of Theorem 1 is along the same lines as the argument in [4].

Proof (Theorem 1). Let $\{x_n\}$ be a sequence of unit vectors such that $\|T^*x_n - \zeta x_n\| \rightarrow 0$ ($n \rightarrow \infty$). Since $\alpha_k \notin \sigma(T) \subset X$ for each k , $T_k = r_k(T - \alpha_k I)^{-1}$ exists as a bounded operator for each k and $\|T_k^*x_n - \zeta_k x_n\| \rightarrow 0$ ($n \rightarrow \infty$), where $\zeta_k = r_k(\zeta - \alpha_k)^{-1}$ (see [9: Lemma 4]). By the growth condition (3)

$$\|T_k\| = r_k \|(T - \alpha_k I)^{-1}\| \leq r_k/d(\alpha_k, X) \leq 1,$$

and so T_k is a contraction. Let U_k be a unitary dilation of T_k . Then

$$\begin{aligned} \|(T_k - \zeta_k I)x_n\|^2 &= \|P(U_k - \zeta_k I)x_n\|^2 \leq \|(U_k - \zeta_k I)x_n\|^2 = \|(U_k - \zeta_k I)^*x_n\|^2 = \\ &= 1 - |\zeta_k|^2 - 2 \operatorname{Re} \zeta_k \langle (T_k - \zeta_k I)^*x_n, x_n \rangle \leq 1 - |\zeta_k|^2 + 2|\zeta_k| \|(T_k - \zeta_k I)^*x_n\|. \end{aligned}$$

Since $T_k - \zeta_k I = r_k(T - \alpha_k I)^{-1}(\zeta I - T)(\zeta - \alpha_k)^{-1}$, we have

$$\begin{aligned} \|(T - \zeta I)x_n\| &\leq r_k^{-1} |\zeta - \alpha_k| \|T - \alpha_k I\| \|(T_k - \zeta_k I)x_n\| \\ &\leq r_k^{-1} |\zeta - \alpha_k| (\|T\| + |\alpha_k|) [1 - |\zeta_k|^2 + 2|\zeta_k| \|(T_k - \zeta_k I)^*x_n\|]^{1/2} \leq \\ &\leq K \cdot [r_k^{-1} |\zeta - \alpha_k|]^2 - 1 + 2r_k^{-1} |\zeta - \alpha_k| \|(T_k - \zeta_k I)^*x_n\|^{1/2} \end{aligned}$$

for some constant $K > 0$. (Note that $\{\alpha_k\}$ is bounded.) Let $\varepsilon > 0$ be given. Then by condition A_X there exists a positive integer k such that

$$0 < (r_k^{-1} |\zeta - \alpha_k|)^2 - 1 < \left(\frac{\varepsilon}{2K}\right)^2.$$

For such a k (fixed), there exists an integer N such that

$$2r_k^{-1} |\zeta - \alpha_k| \|(T_k - \zeta_k I)^* x_n\| < \left(\frac{\varepsilon}{2K}\right)^2 \quad (n > N),$$

because $\|(T_k - \zeta_k I)^* x_n\| \rightarrow 0$ ($n \rightarrow \infty$) and $\{r_k^{-1} |\zeta - \alpha_k|\}$ is bounded. Thus we have

$$\|(T - \zeta I)x_n\| < \varepsilon \quad \text{for } n > N.$$

$\varepsilon > 0$ being arbitrary, $Tx_n - \zeta x_n \rightarrow 0$ ($n \rightarrow \infty$). Hence we have $A_{\zeta}[T^*] \subset A_{\zeta}[T]$. Since the argument is symmetric, relation (2) holds.

We note here that Theorem 1 covers the results of [9, § 3]. A point $\zeta \in X$ is called a semi-bare point [9] if there is a circle through ζ such that no points of X lie inside this circle. This notion is a generalization of bare point of a compact convex set in the plane. If $\zeta \in X$ is a semi-bare point, then ζ satisfies condition A_X . In fact, there exists an $\alpha_0 \notin X$ such that $|\zeta - \alpha_0| = d(\alpha_0, X)$ by the definition of semi-bare point. Let $\{\alpha_n\}$ be a sequence of complex numbers lying on the line segment $\overline{\alpha_0 \zeta}$ such that

$$\frac{1}{(n+1)^{1/2}} < |\alpha_n - \zeta| < \frac{1}{n^{1/2}}.$$

Then $D_n = \{\alpha: |\alpha - \alpha_n| < r_n\}$ with $r_n = |\alpha_n - \zeta| - \frac{1}{(n+1)^{1/2}}$ is contained in the complement of X for each n , and $\alpha_n \rightarrow \zeta$, $r_n^{-1} |\alpha_n - \zeta| \rightarrow 1$ as $n \rightarrow \infty$. By this consideration we have the following corollary [9: Theorems 2 and 3].

Corollary 2. *If T is a hyponormal operator, then the set of semi-bare points of the spectrum $\sigma(T)$ of T does not intersect the residual spectrum of T . In addition, (2) is satisfied for any semi-bare point ζ of $\sigma(T)$.*

Of course, Theorem 1 contains the result by SCHREIBER [7], and Corollary 1 in [3] can be generalized to the case of approximate point spectrum. We have (corresponding to Corollary 1 in [3]) the following

Corollary 3. *Let T satisfy (3) for $X = \sigma(T)$ and let the entire boundary of $\sigma(T)$ be a convex simple closed curve C . Then each $\zeta \in C$ satisfies (2).*

In fact, each point $\zeta \in C$ is a semi-bare point of $\sigma(T)$.

T. YOSHINO kindly communicated the following result. It is published here with his permission.

Corollary 4. *Let T be a hyponormal operator such that $T = A + C$, where C is a compact operator and the spectrum of A lies on a straight line L . Then T is a normal operator.*

C. PUTNAM pointed out to the author (in a private communication) that this can be generalized to a more general result, but it is not in our context.

Proof. We can decompose T into $T = T_1 \oplus T_2$ where T_1 is normal and the point spectrum of T_2 is void [10, Lemma 1]. By [10, Lemma 2], each point of the continuous spectrum of T_2 belongs to the spectrum $\sigma(A)$ of A . By Corollary 2, every semi-bare point of the spectrum $\sigma(T)$ of T_2 is contained in the continuous spectrum of T_2 (note that the point spectrum of T_2 is void). Thus $\sigma(T_2)$ lies on L . Hence T_2 is normal, because a hyponormal operator with the spectrum on the real axis is self-adjoint (see [2, Theorem 1]).

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