N^p-operators and semi-Carleman operators

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1. Let (X, μ) be a measure space, E a Banach space, and let p and p' be the usual conjugate numbers with $1 < p < +\infty$, that is $1/p + 1/p' = 1$. Let $L^p(X, \mu; E)$ be the Banach space of all equivalent classes of μ-strongly measurable E-valued functions K such that $\|K\|^p = \int_X \|K(x)\|^p d\mu < +\infty$.

Operators of the type $T: L^p(X, \mu) \to E$, which can be represented by a unique K in $L^{p'}(X, \mu; E)$ in the following way: $Tg = \int_X g(x)K(x)d\mu$ were considered by A. PERSSON. In [3] he showed that these are operators of type N^p which are also known as right p-nuclear operators. (See [1], Théorème 6.) The author proved in [7] that if E is the strong dual of some Banach space F such that either E is separable or reflexive, then T is the adjoint of an operator $S: F \to L^{p'}(X, \mu)$ such that $|Sf(x)| \leqq \gamma(x)\|f\|$ a.e. for some non-negative γ in $L^{p'}(X, \mu)$. In section 2 of this note we give a new characterization of this class of N^p-operators without referring to their adjoints. A necessary and sufficient condition for T to be of this class is that $\|Tg\| \leqq \int_X \gamma(x)|f(x)|d\mu$ for some non-negative γ in $L^{p'}(X, \mu)$ and for all g in $L^p(X, \mu)$. In section 3, we apply our results to Hilbert spaces. We first give two characterizations of Hilbert—Schmidt class operators, and then obtain a characterization of the semi-Carleman operators introduced by M. SCHREIBER [4]. Finally, we show that the Korotkov theorem for Carleman operators ([2], Theorem 1) remains valid even in nonseparable Hilbert space.

2. Throughout this section, all operators are bounded.

Theorem 2.1. *Let E be a Banach space such that either E has a separable strong dual E' or E is reflexive. For operators $T: L^p(X, \mu) \to E'$ with $1 < p < +\infty$ the following are equivalent:*

(i) *There exists a unique K in $L^{p'}(X, \mu; E')$ such that $Tg = \int_X g(x)K(x)d\mu$ for all g in $L^p(X, \mu)$.*

(ii) *There exists some non-negative γ in $L^{p'}(X, \mu)$ such that $\|Tg\| \leqq \int_X \gamma(x)|g(x)|d\mu$ for all g in $L^p(X, \mu)$.*

Note. The implication (i)⇒(ii) is trivial, as one may take $\gamma(x)=\|K(x)\|$. Moreover, the uniqueness of K in (i) is clear. For if there were some K and K' in $L^{p'}(X, \mu;\ E')$ such that $Tg=\int_X g(x)K(x)d\mu=\int_X g(x)K'(x)d\mu$ for all g in $L^p(X,\mu)$, then, in particular, $\int_A K(x)d\mu=\int_A K'(x)d\mu$ for all measurable set A with finite measure. Because the supports of K and K' are σ-finite measurable sets, we have therefore $K=K'$ in $L^p(X, \mu;\ E')$.

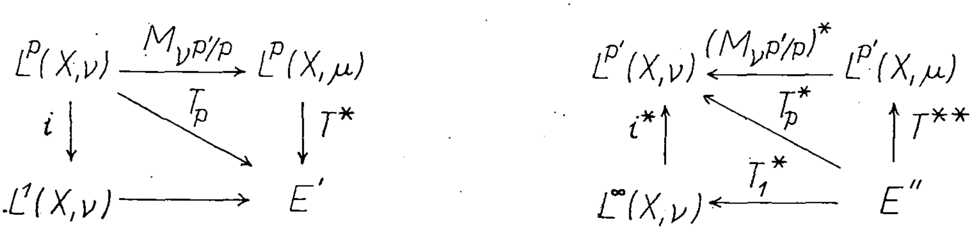
Theorem 2. 1 follows from Theorem 2 of [7] and the following lemma which may have some interest in its own right.

Lemma 2. 1. *Let E be a Banach space. Let $T^*: L^p(X, \mu)\to E'$ be the adjoint of $T: E\to L^{p'}(X, \mu)$, and let $\gamma\in L^{p'}(X, \mu)$, $\gamma\geqq 0$. Then the following are equivalent:*

(i) $|Tf(x)|\leqq\gamma(x)\|f\|$ *a.e. for f in E.*

(ii) $\|T^*g\|\leqq\int_X |g(x)|\gamma(x)d\mu$ *for all g in $L^p(X, \mu)$.*

Proof. *Case* 1: $\gamma(x)>0$ *a.e.* Form the finite measure space (X, ν) where $d\nu=\gamma^{p'}d\mu$. Let $M_\gamma: L^{p'}(X, \nu)\to L^{p'}(X, \mu)$, and $M_{\gamma^{p'/p}}: L^p(X, \nu)\to L^p(X, \mu)$ be the multiplication by γ and $\gamma^{p'/p}$, respectively. That is, $M_\gamma(g)=\gamma\cdot g$ and $M_{\gamma^{p'/p}}(h)=$ $=\gamma^{p'/p}\cdot h$ for g in $L^{p'}(X, \nu)$ and h in $L^p(X, \nu)$. Beause $\gamma(x)<0$ a.e., M_γ and $M_{\gamma^{p'/p}}$ are linear isomorphisms (onto), and $M_\gamma^{-1}=M_{\gamma^{-1}}$, $(M_{\gamma^{p'/p}})^{-1}=M_{\gamma^{-p'/p}}$. A simple computation shows that $(M_{\gamma^{p'/p}})^*=(M_\gamma)^{-1}$, hence $M_\gamma^*=M_{\gamma^{p'/p}}$. We now prove (ii)⇒(i). Write $T_p=T^*\circ M_{\gamma^{p'/p}}$. Then $T_p: L^p(X, \nu)\to E'$, and $\|T_pf\|\leqq\int_X |M_{\gamma^{p'/p}}(f)|\cdot\gamma d\mu$. Hence $\|T_pf\|\leqq\int_X |f(x)|d\nu=\|f\|_1$, where $\|\cdot\|_1$ denotes the L^1-norm of f. Since $L^p(X, \nu)$ is dense in $L^1(X, \nu)$, we can extend T_p to the whole of $L^1(X, \nu)$ without increasing its norm. Let $T_1: L^1(X, \nu)\to E'$ be the extension of T_p. Then $\|T_1\|\leqq 1$. We have the first one of the following commutative diagrams, from which the second one derives by taking adjoints:



Here i and i_λ are the natural embeddings, and $\|T_1^*\|=\|T_1\|\leqq 1$. If f in E, then $T^{**}f=Tf$. (Here we have identified E with a subset of E^{**} via the natural embedding.)

Therefore $i^*T_1^*(f)=(M_{\gamma^{p'/p}})^*T(f)=M_{\gamma^{-1}}(Tf)$. Hence $\|T_1^*f\|_\infty \leqq \|f\|$; it follows that $|T_1^*f(x)| \leqq \|f\|$ a.e. But $T_1^*f(x) = M_{\gamma^{-1}}Tf(x) = \gamma^{-1}(x)\cdot Tf(x)$. Therefore $|\gamma^{-1}(x)\cdot Tf(x)| \leqq \|f\|$ a.e. Hence $|Tf(x)|\leqq\gamma(x)\|f\|$ a.e. for f in E. This completes the proof of the implication (ii)$\Rightarrow$(i). The proof of (i)$\Rightarrow$(ii) is similar. We first consider the mapping $S_{p'}:E\to L^p(X, \nu)$ defined by $S_{p'}=M_{\gamma^{-1}}\circ T$. Then $|S_{p'}f(x)|\leqq\|f\|$ a.e. Let $i:L^\infty(X, \nu)\to L^{p'}(X, \nu)$ be the injection. Then $S_{p'}$ factors as $S_{p'}=i\circ S'_\infty$ where $S_\infty:E\to L^\infty(X, \nu)$ and $\|S_\infty f\|_\infty\leqq\|f\|$ where $\|\cdot\|_\infty$ denotes the L^∞-norm. Hence $\|S_\infty\|\leqq 1$. Therefore $S_\infty^*:M(X, \nu)\to E'$ is also a contraction where $M(X, \nu)$ is the dual of $L^\infty(X, \nu)$. It is clear that $i^*:L^p(X, \nu)\to M(X, \nu)$ is the natural injection which maps g into the finite measure (complex) $g d\nu$ for g in $L^p(X, \nu)$. Hence $\|S_\infty^*\circ i^*g\|\leqq\|i^*g\|$, and $\|i^*g\|=\int_X |g|d\nu$ for g in $L^p(X, \nu)$. Moreover, since $i\circ S_\infty=S_{p'}=M_{\gamma^{-1}}\circ T$, then $S_\infty^*\circ i^*=S_{p'}^*=T^*\circ(M_{\gamma^{-1}})^*=T^*\circ M_{\gamma^{p'/p}}$. It follows that $\|T^*\circ M_{\gamma^{p'/p}}g\|\leqq\int_X |g|d\nu$ for g in $L^p(X, \nu)$. If g is in $L^p(X, \mu)$, write $g=M_{\gamma^{p'/p}}(M_{\gamma^{p'/p}}g)$. Then $\|T^*g\|\leqq$ $\leqq\int_X |M_{\gamma^{-p'/p}}(g)|d\nu=\int_X \gamma(x)|g(x)|d\mu$. This proves (i)$\Rightarrow$(ii).

Case 2: *γ vanishes on a set of positive measure.* Let $Y=\{x;\ \gamma(x)>0\}$, and let (Y, μ) be the measure space obtained by restricting μ to Y. Let $j:L^{p'}(Y, \mu)\to L^{p'}(X, \mu)$ be the natural embedding. Then $j^*:L^p(X, \mu)\to L^p(Y, \mu)$ is the projection $g\to\chi_Y g$ where χ_Y is the characteristic function of Y. Then the operator T factors as $E\xrightarrow{T_Y} L^p(Y, \mu)\xrightarrow{j} L^p(X, \mu)$ if and only if T^* factors as $L^{p'}(X, \mu)\xrightarrow{j^*} L^{p'}(Y, \mu)\xrightarrow{T_Y^*} E'$. Now we apply the implication $(i)\Rightarrow$(ii) to the operators T_Y and T_Y^*, and complete the proof.

Proof of Theorem 2. 1. We only need to prove that (ii) implies (i). Let $S:E\to L^{p'}(X, \mu)$ be the restriction of $T^*:E''\to L^{p'}(X, \mu)$ to E. Then $T=S^*$. By Lemma 2. 1, we have $|Sf(x)|\leqq\gamma(x)\|f\|$ a.e. for f in E. By Theorem 2 of [7], we have $Tg=S^*g=\int_X K(x)g(x)d\mu$ for a unique K in $L^{p'}(X, \mu;\ E')$.

Remark. We note that, in Theorem 2. 1, the existence of K does not depend upon the choice of those non-negative γ such that $\|Tg\|\leqq\int_X \gamma(x)|g(x)|d\mu$ for g in $L^p(X, \mu)$. The following lemma asserts that the function $\|K(\cdot)\|$ is the infimum of all those γ in the language of lattice theory. That is, $\|K(\cdot)\| = \Lambda\{\gamma\in L^{p'}(X, \mu);\ \|Tg\|\leqq\int_X \gamma(x)|g(x)|d\mu$ for all g in $L^p(X, \mu)\}$.

Lemma 2. 2. *Let E be a Banach space. Let K be in $L^{p'}(X, \mu;\ E)$ and let γ be non-negative element in $L^{p'}(X, \mu)$ such that $\left\|\int_X K(x)g(x)d\mu\right\|\leqq\int_X \gamma(x)|g(x)|d\mu$ for all g in $L^p(X, \mu)$. Then $\|K(x)\|\leqq\gamma(x)$ a.e.*

Proof. Let $S:L^p(X,\mu)\to E$ be defined by $Sg=\int_X K(x)g(x)d\mu$. Then S is a bounded operator, and $S^*:E'\to L^{p'}(X,\mu)$ is given by $S^*f'(x)=\langle f',K(x)\rangle$ a.e. Furthermore, the proof for (ii)⇒(i) of Lemma 2. 1 proves that $|S^*f'(x)|\leqq\gamma(x)\|f'\|$ a.e. where the exceptional set of measure zero may depend upon f'. Hence $|\langle f',K(x)\rangle|\leqq\gamma(x)\|f'\|$ a.e. for f' in E'. Let N be the μ-null set such that $K(X\setminus N)$ is contained in a separable subset of E. Let $\{f_1,f_2,\dots,f_n,\dots\}$ be a countable dense subset of this subset of E. Let $\{f_1',f_2',\dots,f_n',\dots\}$ be the subset of E' such that $\|f_j'\|=1$ and $|\langle f_j',f_j\rangle|=\|f_j\|$ for each j. Then, if x is not in N, we have $\|K(x)\|=$ $=\sup_j|\langle f_j',K(x)\rangle|$. Let N_j be the μ-null such that $|\langle f_j',K(x)\rangle|\leqq\gamma(x)$ for all x not in N_j. Let $A=N\cup\left(\bigcup_{j=1}^{\infty}N_j\right)$. Then A is also a μ-null, and $\|K(x)\|\leqq\gamma(x)$ for all x not in A. This proves the lemma.

3. Let H be a Hilbert space. Let $S:H\to L^2(X,\mu)$ be a Hilbert—Schmidt class operator. For any orthonormal basis $\{f_\lambda\}$ of H, $\sum_\lambda\|Sf_\lambda\|^2$ is finite. There are at most countably many non-vanishing $\|Sf_\lambda\|^2$ in the above sum, say $Sf_{\lambda_j}\neq0$ $(j=1,2,3,\dots)$. Hence $\sum_{j=1}^{\infty}|Sf_{\lambda_j}(x)|^2<+\infty$ a.e. Let $K(x)=\sum_{j=1}^{\infty}Sf_{\lambda_j}(x)f_{\lambda_j}$. Then K is a strongly μ-measurable H-valued function such that $\int_X\|K(x)\|^2d\mu=\sum_{j=1}^{\infty}\|Sf_{\lambda_j}\|^2=\sum_\lambda\|Sf_\lambda\|^2=\|S\|_2^2$, where $\|S\|_2$ denotes the Hilbert—Schmidt norm of S. Furthermore, $Sf_{\lambda_j}(x)=$ $=\langle f_{\lambda_j},K(x)\rangle$ and hence $Sf(x)=\langle f,K(x)\rangle$ a.e. for f in H. Conversely, if K in $L^2(X,\mu;H)$ and $S:H\to L^2(X,\mu)$ is defined by $Sf(x)=\langle f,K(x)\rangle$ a. e. then it is clear that S is of Hilbert—Schmidt class with Hilbert—Schmidt norm $\|K\|$. This shows that every Hilbert—Schmidt class operator $S:H\to L^2(X,\mu)$ is of the form $Sf(x)=$ $=\langle f,K(x)\rangle$ a.e. for a unique K in $L^2(X,\mu;H)$. The above argument can also be found, for example, in [6], 2. 2 (1); we include it here for a later reference. The following characterization for Hilbert—Schmidt class operators first appeared in PERSSON's article ([3], Theorem 3) as a special case of his main result. It is also included in ([7], Corollary 3 and its following remark).

However, the following version is due to WEIDMANN ([6], 2. 10. Korollar) for separable Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space. For a bounded operator T: $H\to$ $\to L^2(X,\mu)$, the following are equivalent:*

(i) *T is of Hilbert—Schmidt class.*

(ii) *$|Tf(x)|\leqq\gamma(x)\|f\|$ a.e. for some non-negative γ in $L^2(X,\mu)$.*

(iii) *$Tf(x)=\langle f,K(x)\rangle$ a.e. for a unique K in $L^2(X,\mu;H)$.*

Moreover, $\|T\|_2=\|K\|$, where $\|T\|_2$ denotes the Hilbert—Schmidt norm of T, and $\|K\|$ denotes the norm of K in $L^2(X,\mu;H)$.

Proof. The argument given at the beginning of this section shows that (i) and (iii) are equivalent and $\|T\|_2=\|K\|$. By Theorem 1 of [7] we see that (ii) and (iii) are equivalent.

Dual to Theorem 3. 1, we have the following

Theorem 3. 2. *Let H be a Hilbert space. For a bounded operator $S:L^2(X,\mu)\to H$ the following are equivalent:*

(i) *S is of Hilbert—Schmidt class.*

(ii) *$\|Sg\|\leqq\int_X \gamma(x)\,|g(x)|d\mu$ for some non-negative γ in $L^2(X,\mu)$.*

(iii) *$Sg=\int_X K(x)g(x)d\mu$ for a unique K in $L^2(X,\mu;\,H)$.*

Moreover, $\|S\|_2=\|K\|$.

Proof. S is of Hilbert—Schmidt class if and only if $S^*:H\to L^2(X,\mu)$ is of Hilbert—Schmidt class. This is so if and only if $|S^*f(x)|\leqq\gamma(x)\|f\|$ a.e. for some $\gamma\geqq 0$ in $L^2(X,\mu)$. By Lemma 2. 1, the above inequality holds if and only if $\|Sg\|\leqq\leqq\int_X\gamma(x)|g(x)|d\mu$. Hence (i) and (ii) are equivalent. (ii) and (iii) are equivalent by Theorem 2. 1. Furthermore, from Theorem 3. 1, we have $\|S^*\|_2=\|K\|$, but $\|S\|_2==\|S^*\|_2$. Hence $\|S\|_2=\|K\|$.

We now turn our attention to operators defined on a linear manifold of Hilbert space. Let H be a Hilbert space, and let K be a strongly μ-measurable H-valued function defined almost everywhere on X. Let $\mathfrak{D}=\{f\in H;\ \langle f, K(\cdot)\rangle\in L^2(X,\mu)\}$. Then $\mathfrak{D}$ is a linear manifold of H, but not necessarily dense in H. Let $\hat{\mathfrak{D}}=\{g\in L^2(X,\mu):\int_X\|K(x)\|\ |g(x)|d\mu<+\infty\}$. Then $\hat{\mathfrak{D}}$ is a dense linear manifold of $L^2(X,\mu)$ (cf. [4]).

Notice that $\hat{\mathfrak{D}}=\{g\in L^2(X,\mu);\ gK$ is Bochner integrable$\}$. Moreover, if $\hat{\mathfrak{D}}=L^2(X,\mu)$, then K is necessary in $L^2(X,\mu;\,H)$.

Following J. Weidmann [6] we call an operator $T:\mathfrak{D}_T\to L^2(X,\mu)$ a Carleman operator, if its domain $\mathfrak{D}_T$ is contained in $\mathfrak{D}$ and it can be written as $Tf(x)=\langle f, K(x)\rangle$ a.e. for f in $\mathfrak{D}_T$. An operator $S:\hat{\mathfrak{D}}_S\to H$ is called a semi-Carleman operator, if its domain $\hat{\mathfrak{D}}_S$ is contained in $\hat{\mathfrak{D}}$ and it can be written as $Sg=\int_X g(x)K(x)d\mu$ for g in $\hat{\mathfrak{D}}_S$.

We note that, when (X,μ) is σ-finite, and $H=L^2(X,\mu)$, then our definitions for Carleman and semi-Carleman operations coincide with the classical ones ([2] and [4]). For a detailed discussion of this see ([6], Section 5) or ([2], Lemma 1).

Theorem 3. 3. *Let (X,μ) be a σ-finite measure space. Let $S:\hat{\mathfrak{D}}_S\to H$ be an operator with dense domain $\hat{\mathfrak{D}}_S$ in $L^2(X,\mu)$. The following are equivalent:*

(i) *S is a semi-Carleman operator.*

(ii) *There exists a measurable function* γ *such that* $0 \leqq \gamma(x) < +\infty$ *a.e.*, $\hat{\mathfrak{D}}_S \subset \{g \in L^2 : \int_X \gamma(x)\, |g(x)| d\mu < +\infty\}$, *and* $\|Sg\| \leqq \int_X \gamma(x)\, |g(x)| d\mu$ *for all* g *in* $\hat{\mathfrak{D}}_S$.

Proof. The implication (i)$\Rightarrow$(ii) is clear. We now prove (ii)$\Rightarrow$(i). Write $X = \bigcup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$ and $\mu(A_n) < +\infty$ for all n. Let $X_n = \{x \in A_n;\ \gamma(x) \leqq n\}$ for $n = 1, 2, \ldots$. Then $X_n \subset X_{n+1}$, $\mu(X_n) < +\infty$ and $\mu(X \setminus \bigcup_n X_n) = 0$. Let μ_n be the restriction of μ to X_n, let $\hat{\mathfrak{D}}_n = \hat{\mathfrak{D}}_S \cap L^2(X_n, \mu_n)$, $\gamma_n = \chi_{X_n}\gamma$. Then γ_n is in $L^2(X_n, \mu_n)$, and $\hat{\mathfrak{D}}_n$ is dense in $L^2(X_n, \mu_n)$. Consider $S_n : \hat{\mathfrak{D}}_n \to H$; the restriction of S to $\hat{\mathfrak{D}}_n$. We have $\|S_n g\| \leqq \leqq \int_{X_n} \gamma_n(x)\, |g(x)| d\mu_n$ for g in $\hat{\mathfrak{D}}_n$. Then S_n admits a unique bounded extension to $L^2(X_n, \mu_n)$ which is also denoted by S_n. Moreover, the inequality $\|S_n g\| \leqq \leqq \int_X \gamma_n(x)\, |g(x)| d\mu_n$ holds for all g in $L^2(X_n, \mu_n)$. Therefore, by Theorem 3. 2 $S_n g = = \int_X g(x) K_n(x) d\mu_n$ for a unique K_n' in $L^2(X_n, \mu_n; H)$. By Lemma 2. 2 $\|K_n'(x)\| \leqq \gamma_n(x)$ a.e. Note that S_{n+1} extends S_n, using the uniqueness assertion once more we have $K_{n+1}'(x) = K_n'(x)$ a.e. on X_n. We now define K_n almost everywhere on X by putting $K_n(x) = K_n'(x)$ a.e. on X_n and $K_n(x) = 0$ for x not in X_n. Then K_n is μ-strongly measurable. Since $K_{n+1}(x) = K_n(x)$ a.e. on X_n, then $\lim_{n\to\infty} K_n(x)$ exists almost everywhere. Let $K(x) = \lim_{n\to\infty} K_n(x)$, then K is defined almost everywhere on X into H and K is also μ-strongly measurable. Moreover, $\|K(x)\| = \lim_{n\to\infty} \|K_n(x)\| \leqq \lim_{n\to\infty} \gamma_n(x) = \gamma(x)$ a.e. Hence $\int_X |g(x)|\, \|K(x)\| d\mu \leqq \int_X |g(x)| \gamma(x) d\mu < +\infty$ for all g in $\hat{\mathfrak{D}}_S$. Thus the integral $\int_X g(x) K(x) d\mu$ exists for g in $\hat{\mathfrak{D}}_S$. We have $\hat{\mathfrak{D}}_S \subset \{g \in L^2(X, \mu);\ \int |g(x)|\, \|K(x)\| d\mu < +\infty\}$. We want to show that $Sg = \int_X g(x) K(x) d\mu$ for g in $\hat{\mathfrak{D}}_S$. To see this, we let $g_n = \chi_{X_n} g$. Then $g_n(x) \to g(x)$ a.e. and $g_n \in L^2(X_n, \mu_n)$. $\|Sg - Sg_n\| \leqq \int_X \gamma(x)\, |g_n(x) - g(x)|\, d\mu \to 0$, by dominated convergence. But $Sg_n = S_n g_n = \int_{X_n} K_n(x) g_n(x) d\mu_n = \int_X K(x) g_n(x) d\mu$. On the other hand $\left\| \int_X K(x) g(x) d\mu - \int_X K(x) g_n(x) d\mu \right\| \leqq \int_X \|K(x)\|\, |g(x) - g_n(x)| d\mu \leqq \leqq \int_X \gamma(x)\, |g(x) - g_n(x)|\, d\mu \to 0$. Therefore $Sg = \int_X g(x) K(x) d\mu$ for g in $\hat{\mathfrak{D}}_S$. This completes the proof.

In 1965, V. B Korotkov gave a characterization for a Carleman operator on separable L^2-space which is what he called an integral operator of Carleman type (cf. [2], Theorem 1). His proof is based on the Dunford-Pettis Theorem. Recently, M. Schreiber and Gy. Targonski also obtained a new characterization for Carleman operators (cf. [5], Theorem 2. 1). However, J. P. Williams shows that the

Schreiber—Targonski theorem is a consequence of the Korotkov theorem (private communication). (See also [6], Satz 2. 11.) Using our result, we can prove that the Korotkov Theorem remains valid without the separability assumption on the Hilbert spaces.

Theorem 3. 4 (Korotkov [2], Theorem 1). *Let (X, μ) be a σ-finite measure space. Let $T:\mathfrak{D}_T \to L^2(X, \mu)$ be an operator with dense domain $\mathfrak{D}_T$ in a Hilbert space H. The following are equivalent conditions:*

(i) *T is a Carleman operator.*

(ii) *There exists a non-negative measurable function γ such that $\gamma(x) < +\infty$ a.e. and $|Tf(x)| \leqq \gamma(x)\|f\|$ a.e. for f in $\mathfrak{D}_T$.*

Proof. (i) clearly implies (ii). To prove (ii)$\Rightarrow$(i), we write $X = \bigcup_{n=1}^{\infty} A_n$, with $A_n \subset A_{n+1}$ and each A_n of finite measure. Let $X_n = \{x \in A_n;\ \gamma(x) \leqq n\}$. Then $\mu\left(X \setminus \bigcup_{n=1}^{\infty} X_n\right) = 0$, and $X_n \subset X_{n+1}$ and each X_n has finite measure. Let $E_n: L^2(X, \mu) \to L^2(X, \mu)$ be the projection on $L^2(X_n, \mu)$. Then $E_n \to 1$ strongly. Let $\gamma_n = \chi_{X_n}\gamma$, $\mu_n = \mu|_{X_n}$. Then γ_n in $L^2(X_n, \mu_n)$. Consider $E_n T: \mathfrak{D}_T \to L^2(X, \mu)$. Let $j: L^2(X_n, \mu_n) \to L^2(X, \mu)$ be the natural embedding. Then $E_n T$ factors as $\mathfrak{D}_T \xrightarrow{T_n} L^2(X_n, \mu_n) \xrightarrow{j} L^2(X, \mu)$ where $|T_n f(x)| \leqq \leqq \gamma_n(x)\|f\|$ a. e. (μ_n). T_n admits a unique bounded extension to H which is again written as T_n. By a standard density argument one can show that the extension T_n also has the property that $|T_n f(x)| \leqq \gamma_n(x)\|f\|$ a.e. (μ_n) for f in H. By Theorem 3. 1 T_n is of Hilbert—Schmidt class, and there is a unique K_n in $L^2(X_n, \mu_n;\ H)$ such that $T_n f(x) = \langle f, K_n'(x)\rangle$ a.e. By uniqueness again, we have $K_{n+1}'(x) = K_n'(x)$ a.e. on X_n. Let $K_n(x) = K_n'(x)$ a.e. on X_n and $K_n(x) = 0$ for x not in X_n. Then each K_n is strongly μ-measurable H-valued. Let $K(x) = \lim_{i\to\infty} K_n(x)$ a.e. Then K defines almost everywhere on X and is μ-measurable. Moreover $K(x) = K_n(x)$ a.e. on X_n. If f in $\mathfrak{D}_T$, then $Tf = \lim_{i\to\infty} E_n Tf$. But $E_n Tf = j \cdot T_n f$, so $(E_n T)f(x) = (T_n f)(x) = f, K_n(x)$ a. e. Therefore $Tf(x) = \lim_{i\to\infty} E_{n_i} Tf(x) = \lim_{i\to\infty}\langle f, K_{n_i}(x)\rangle = \langle f, K(x)\rangle$ a.e. This completes the proof.

4. Concluding remark. In the definition of a semi-Carleman operator, if we enlarge the linear manifold $\hat{\mathfrak{D}}$ to the linear manifold of $L^2(X, \mu)$ consisting of all g such that the H-valued function $x \to g(x)K(x)$ is weakly integrable in the sense of Pettis, where K is a μ-strongly measurable H-valued function. We may call an operator $T: \hat{\mathfrak{D}}_T \to H$ a weak semi-Carleman operator, if its domain $\hat{\mathfrak{D}}_T$ is contained in $\hat{\mathfrak{D}}$ and it can be written as $Tg = \int_X g(x)K(x)\,d\mu$ for g in $\hat{\mathfrak{D}}_T$, where the integral is the weak integral in the sense of Pettis. It is easy to see that, if $A: H \to \to L^2(X, \mu)$ is an everywhere defined Carleman operator (hence bounded), then

$A^*: L^2(X, \mu) \to H$ is a weak semi-Carleman operator. More than this, one can easily show that the adjoint of a densely defined Carleman operator is a closed extension of a weak semi-Carleman operator. It follows that the conditions $|Af(x)| \leqq \gamma(x)\|f\|$ and $\|A^*g\| \leqq \int_X \gamma(x)|g(x)|d\mu$, for some nonnegative measurable γ are not equivalent for the Carleman operator A. It would be interesting to give a characterization for a weak semi-Carleman operator.

Using theory of semi-ordered spaces, S. I. ŽDANOV (cf. [8], proof of Theorem 1) proved that the Korotkov inequality $|Tf(x)| \leqq \gamma(x)\|f\|$ a.e. is equivalent to that T maps every null sequence of vectors $\{f_n\}_{n=1}^{\infty}$ in H into a sequence $\{Tf_n\}_{n=1}^{\infty}$ in $L^2(X, \mu)$ such that $Tf_n(x) \to 0$ a.e. For a complete elementary proof of this see WEIDMANN ([6], Satz 2. 12). We do not know the answer to the following question:

What is the condition corresponding to the Ždanov theorem for a semi-Carleman operator and a Carleman operator respectively?

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