Some theorems on unitary ϱ-dilations of Sz.-Nagy and Foiaş

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Introduction. SZ.-NAGY and FOIAŞ introduced, for each fixed $\varrho>0$, the class C_ϱ of operators T on a given complex Hilbert space H for which there exist a Hilbert space K containing H as a subspace and a unitary operator U on K satisfying the following relation:

$$T^n = \varrho \cdot PU^n \qquad (n = 1, 2, \dots) \tag{1}$$

where P is the orthogonal projection of K on H; this unitary operator U is called a unitary ϱ-dilation of T.

It is well known that $C_1 = \{T: \|T\| \leqq 1\}$ ([7]) and that $C_2 = \{T: w(T) \leqq 1\}$ ([1]), where $w(T)$ denotes the numerical radius of T i.e.

$$w(T) = \sup |(Th, h)| \quad \text{for} \quad h \in H, \|h\| = 1. \tag{2}$$

SZ.-NAGY and FOIAŞ have characterized C_ϱ for general $\varrho>0$. One of their results is:

Theorem A ([8]). *An operator T on H belongs to the class C_ϱ ($\varrho \geqq 2$) if and only if it satisfies the following conditions:*

$$\|(\mu I - T)^{-1}\| \leqq \frac{1}{|\mu|-1} \begin{cases} \textit{for} \quad 1 < |\mu| < \infty & \textit{if } \varrho = 2, \\ \textit{for} \quad 1 < |\mu| \leqq \dfrac{\varrho-1}{\varrho-2} & \textit{if } \varrho > 2, \end{cases} \tag{$*$}$$

($**$) *T has its spectrum in the closed unit disc.*

In [6] J. A. R. HOLBROOK introduced the functions $w_\varrho(T)$ defined on the space $B(H)$ of all operators on H as follows

$$w_\varrho(T) = \inf\left\{u: u > 0, \ \frac{1}{u} T \in C_\varrho\right\}; \tag{3}$$

in particular, we have $w_2(T) = w(T)$, $w_1(T) = \|T\|$, and

$$C_\varrho = \{T: w_\varrho(T) \leqq 1\}. \tag{4}$$

The following theorem holds:

Theorem B ([6]). *$w_\varrho(T)$ has the following properties:*

(i) $w_\varrho(T)<\infty$;

(ii) $w_\varrho(T)>0$ *unless* $T=0$, *in fact* $w_\varrho(T)\geqq\frac{1}{\varrho}\|T\|$;

(iii) $w_\varrho(zT)=|z|w_\varrho(T)$;

(iv) $w_\varrho(T)$ *is a norm whenever* $0<\varrho\leqq 2$;

(v) $w_\varrho(T)$ *is continuous and non-increasing as a function of* ϱ; *moreover,* $r(T)\leqq w_\varrho(T)$ *for* $\varrho>0$ *and* $\lim_{\varrho\to\infty} w_\varrho(T)=r(T)$, *where* $r(T)$ *is the spectral radius of* T;

(vi) *the "power inequality" holds:* $w_\varrho(T^k)\leqq(w_\varrho(T))^k \qquad (k=1,2,\ldots)$.

In [2] and [8] there are given examples of power bounded operators which are not contained in any of the classes C_ϱ.

1. The theorems and their corollaries

Theorem 1. *If $T^2=T$ and $T\in C_\varrho$, then T is a projection.*

Theorem 2. *If $T^k=T$ for some positive integer $k\geqq 2$ and $T\in C_\varrho$, then T is the direct sum of a zero operator and of a unitary operator, i.e. T is normal and partially isometric.*

Corollary 1 ([4]). *If T is an idempotent operator that satisfies any of the following conditions*

(i) *T is a contraction;*

(ii) *T is a numerical radius contraction* $(w(T)\leqq 1)$,

(iii) *T has equal norm and spectral radius (normaloid* [5]),

(iv) *T has equal numerical and spectral radius (spectraloid* [5]),

then T is an orthogonal projection.

Corollary 2 ([4]). *If $T^k=T$ for some positive integer $k\geqq 2$ and satisfies any of the conditions* (i)—(iv) *in Corollary* 1, *then T is the direct sum of a zero operator and of a unitary operator, i.e. T is normal and partially isometric.*

Corollary 3. *If $T^k=T$ for some positive integer $k\geqq 2$ and $\|T\|>1$, then T is not contained in any of the classes C_ϱ.*

Corollary 3 gives another simple examples of power bounded operators which are not contained in any of the classes C_ϱ.

Proof of Theorem 1. By the idempotency of T, $R(T)$ (the range of T) coincides with null space of $I-T$, so that $R(T)$ is a closed subspace of H. Let P_1 and P_2 denote the orthogonal projections of H onto $R(T)$ and $R(T)^\perp$, respectively.

We consider the matrix of T with respect to the decomposition $H=R(T)\oplus R(T)^{\perp}$ i.e.

$$T=\begin{pmatrix}P_1TP_1 & P_1TP_2\\ P_2TP_1 & P_2TP_2\end{pmatrix}=\begin{pmatrix}I & S\\ O & O\end{pmatrix},\quad (\mu I-T)^{-1}=\begin{pmatrix}\frac{1}{\mu-1}I & \frac{1}{\mu(\mu-1)}S\\ O & \frac{1}{\mu}I\end{pmatrix}.$$

We suppose that T is not a projection, that is, $S\neq 0$. Then

$$\|(\mu I-T)^{-1}\|=\sqrt{\frac{1}{|\mu-1|^2}+\frac{\|S\|^2}{|\mu(\mu-1)|^2}}>\frac{1}{|\mu-1|};$$

by taking μ real with $1<\mu\leqq\frac{\varrho-1}{\varrho-2}$, we obtain

$$\|(\mu I-T)^{-1}\|>\frac{1}{|\mu-1|}=\frac{1}{|\mu|-1}.$$

Hence T does not satisfy condition (*) for any $\varrho\geqq 2$. Since C_ϱ is a non-decreasing function of ϱ, we have $T\notin C_\varrho$ for any $\varrho>0$. This contradiction proves Theorem 1.

Theorem 3. *If $T^k=T$ for some postive integer $k\geqq 2$ and $T\in C_\varrho$, then T^{k-1} is a projection.*

Proof. We have $T^{2(k-1)}=T^{k-2}T^k=T^{k-2}T^1=T^{k-1}$, which implies that T^{k-1} is an idempotent operator. Hence by (4) and the power inequality for $w_\varrho(T)$ we have $w_\varrho(T^{k-1})\leqq(w_\varrho(T))^{k-1}\leqq 1$ so that $T^{k-1}\in C_\varrho$; thus T^{k-1} is a projection by Theorem 1.

Proof of Theorem 2. It is sufficient to consider the case that $T^k=T$ and $T\in C_\varrho$, where $k\geqq 2$ and $\varrho\geqq 1$. By Theorem 3, $P=T^{k-1}$ is a projection. Set $M=R(P)$. The relation $T=TP=PT$ implies that M reduces T and that T is zero on $M^{\perp}$.

On the other hand, $T_1=T|M$ satisfies $T_1^{k-1}=P|M=I_M$ and $w_\varrho(T_1)\leqq 1$. Thus we have $T_1^{-1}=T_1^{k-2}$. By the power inequality for $w_\varrho(T)$

$$w_\varrho(T_1^{-1})=w_\varrho(T_1^{k-2})\leqq(w_\varrho(T_1))^{k-2}\leqq 1,$$

whence we have $w_\varrho(T_1)\leqq 1$ and $w_\varrho(T_1^{-1})\leqq 1$ for $\varrho\geqq 1$, therefore T_1 is unitary ([9]). Consequently T is the direct sum of zero operator and of a unitary operator, that is to say, T is normal and partially isometric.

Corollaries 1 and 2 follow from Theorems 1 and 2 and from the fact that $w_\varrho(T)$ is a continuous and non-increasing function of ϱ. Corollary 3 is obvious by Theorem 2.

If $T^2=I$ and $T\in C_\varrho$, then T is unitary ([9]). Hence we remark that if $T^2=I$ and $\|T\|>1$, then $T\notin C_\varrho$ for any ϱ, in fact there are given two concrete examples in [2] and [8], which satisfy $T^2=I$ and $T\notin C_\varrho$ for any ϱ.

2. "ϱ-oid" operators

Definition 1 ([3]). An operator T will be called "*ϱ-oid*" if

$$w_\varrho(T^k)=(w_\varrho(T))^k \qquad (k=1, 2, \ldots);$$

1-oid and 2-oid operators are normaloid and spactraloid, respectively ([5]).

Theorem C ([3]), *For each* $\varrho \geqq 1$,

$$w_\varrho(T)=r(T) \quad \textit{if and only if} \quad w_\varrho(T^k)=(w_\varrho(T))^k \qquad (k=1, 2, \ldots).$$

For each $0<\varrho<1$ *there exists no non-zero "ϱ-oid" operator which is included in the class of normaloids* ([3]).

By the power inequality $w_\varrho(T^k)\leqq(w_\varrho(T))^k$ $(k=1, 2, \ldots)$, Theorems 1 and 2, we have the following corollaries.

Corollary 4. *If T is "ϱ-oid" and $T^2=T$, then T is a projection.*

Corollary 5. *If T is "ϱ-oid" and $T^k=T$ for some positive integer $k\geqq 2$, then T is the direct sum of zero and a unitary operator, that is to say, T is normal and partially isometric.*

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