A metric characterization of homogeneous Riemannian manifolds

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Let M be a Riemannian manifold and $\varrho(x, y)$ the infimum of the length of those piecewise C^1-curves which join x, y in M. As well-known ϱ is a distance function on M and the thus *induced metric space* $[M, \varrho]$ is so closely related to the Riemannian manifold that a considerable number of theorems about it can be formulated and proved merely in terms of $[M, \varrho]$. This circumstance can be regarded as the starting point of the theories of H. BUSEMANN and W. RINOW where a metric space is the basic concept and some fundamental properties common to all metric spaces induced by Riemannian or Finsler manifolds are being postulated. Although these theories go beyond the scope of the standard one, e.g. as to differentiability conditions, their exact relation to it is not sufficiently clarified yet. In other words no adequate necessary and sufficient conditions are known which imply that a metric space should be induced by a Riemannian manifold. A partial solution of this problem is presented below, i.e. necessary and sufficient conditions are given for the case of metric spaces induced by homogeneous Riemannian manifolds.

1. Basic concepts and the main result

Some well-known fundamental facts concerning metric spaces induced by C^∞-Riemannian manifolds are summarized here. (For a detailed presentation see [5].)

A metric space is said to be *finitely compact* if any bounded infinite subset has a point of accumulation in it. Metric spaces induced by complete Riemannian manifolds are finitely compact. A locally distance preserving map of the real line into a metric space is called a *geodesic*. The geodesics of a Riemannian manifold which are parametrized in terms of arc length and geodesics of its induced metric space are the same. If a, b, c are distinct points of a metric space $[R, \varrho]$ and $\varrho(a, c)+$ $+\varrho(c, b) = \varrho(a, b)$, then it is said that *c lies between a* and b, in notation: acb. If $A \subset R$ and to any two different points a, b of A there is a $c \in A$ with acb, then A is said to be *convex*. The induced space of a complete Riemannian manifold is convex. A distance preserving map of a compact interval of the real line into a metric space is called a *segment*. If $[R, \varrho]$ is a finitely compact and convex metric space then any two points can be joined by a segment in it. The segments are said to be *locally prolongable* in a finitely compact convex metric space $[R, \varrho]$ if to any $p \in R$ there is such a $\delta_p > 0$ that to any two distinct points a, b in $B(p, \delta_p) = \{x : \varrho(x, p) < \delta_p\}$ there is a $c \in R$ with abc. It is said that the *prolongation of segments is unique* in $[R, \varrho]$ if $x, y, z', z'' \in R$, xyz', xyz' and $\varrho(x, z') = \varrho(x, z'')$ imply $z' = z''$. The above terminology is justified by the fact that the segments of a finitely compact convex metric space are uniquely extendable to geodesics if the preceding two conditions hold. The closed subset $A \subset R$ is called *strictly convex* if it is convex and $a, b, c \in A$, acb imply that $c \in \operatorname{int} A$. The metric space $[R, \varrho]$ is called *regular* if to any $p \in R$ there are such $\varkappa_p, \lambda_p > 0$ that the closed balls $\overline{B(x, \xi)}$ are strictly convex if $x \in B(p, \varkappa_p)$ and $0 < \xi \leqq \lambda_p$. Riemannian manifolds induce regular metric spaces.

The induced metric space of a Finsler manifold can be defined analogously and the above facts generalize to their case as well; see [9]. A connection with the induced metric space peculiar to Riemannian manifolds can be expressed in terms of the metric angle concept. Let a, b, c be points of a metric space $[R, \varrho]$ then there are points A, B, C of the euclidean plane with $\varrho(a, b) = \overline{AB}$, $\varrho(b, c) = \overline{BC}$, $\varrho(c, a) = \overline{CA}$. If $a \neq b, c$, then by the metric angle $\gamma(a; b, c)$ of the triple $\{a, b, c\}$ at a the measure of $\sphericalangle BAC$ is meant. Let $\varphi, \psi : [0, \alpha] \to R$ be continuous curves with $\varphi(0) = \psi(0) = x$ and with such a $0 < \delta \leqq \alpha$ that $\varphi(\tau), \psi(\tau) \neq x$ for $0 < \tau \leqq \delta$. If $\gamma(\varphi, \psi) = \lim_{\tau', \tau'' \to 0} \gamma(x; \varphi(\tau'), \psi(\tau''))$ exists, then this value is called the *metric angle* of φ and ψ at x. If φ, ψ are differentiable curves of a Riemannian manifold then considered as curves of the induced metric space they have a metric angle which is equal to the one which they have as curves of the Riemannian manifold; see [7].

An isometric transformation of a Riemannian manifold is obviously a distance preserving transformation of its induced metric space. The converse of this assertion is a theorem due to S. B. Myers and N. Steenrod (see [6]).

Let $\Phi : R^1 \times S \to S$ be a continuous 1-parameter group of transformations of the topological space S, then the continuous curve $\varphi : R^1 \to S$ defined by $\varphi(\tau) = \Phi(\tau, x)$, $\tau \in R^1$ is called the *orbit of Φ starting at $x \in S$*.

The main result of this paper is the following

Theorem 1. *Let $\Gamma : G \times R \to R$ be an effective and transitive transformation group, where R has a distance function ϱ such that the elements of G are distance preserving transformations of $[R, \varrho]$. Assume that*

1. *$[R, \varrho]$ is finitely compact,*
2. *$[R, \varrho]$ is convex,*

3. *the segments are locally prolongable in* $[R, \varrho]$,
4. *the prolongation of segments is unique,*.
5. $[R, \varrho]$ *is regular,*
6. *the orbits of* 1*-parameter groups of distance preserving transformations are rectifiable in* $[R, \varrho]$,
7. *if two such orbits have a point in common then they have a metric angle there.*

Then G with the compact-open tolpogy is a topological group and Γ is a continuous transformation group. The identity component G_0 of G is a Lie group and R has a unique differentiable manifold structure such that $\Gamma_0 : G_0 \times R \to R$, the restriction of Γ, is a transitive differentiable transformation group. There is a unique Riemannian manifold structure on R which has $[R, \varrho]$ as its induced metric space.

Conditions 1—4 have been introduced by H. BUSEMANN [1] as the starting point for his theory of G-spaces.

The proof of the above theorem is carried out in two steps: first a differentiable structure is introduced on R, secondly a Riemannian structure. These two steps are summarized in Theorem 2 and 3. Theorem 1 is a direct consequence of these two theorems.

Conditions 1—7 of Theorem 1 will be generally assumed to hold in what follows. Differentiability will mean C^∞, unless it is not explicitely otherwise stated, although in some cases obviously less would suffice or more could be stated.

2. The introduction of the differentiable structure

The initial step in introducing the differentiable structure of R is the definition of an appropriate topology in the group of distance preserving transformations. This can be done by an obvious application of standard methods (see [5]) by proving

Lemma 2.1. *Let $[R, \varrho]$ be a finitely compact metric space and $\Gamma : G \times R \to R$ an effective transformation group where the elements of G are distance preserving transformations of $[R, \varrho]$, then with the compact-open topology G is a σ-compact group and Γ a toplogical transformation group.*

The next step is to show that the identity component G_0 of G is a Lie group. Owing to a theorem of A. GLEASON and H. YAMABE (see [3], [10]) it suffices to prove that G has no small subgroups. But this is asserted in the following lemma which has been proved already elsewhere (see [8]):

Lemma 2.2. *Let $\Gamma : G \times R \to R$ be an effective transformation group where R has a distance function ϱ such that $[R, \varrho]$ is a finitely compact convex and regular* *metric space in which segments are locally and uniquely prolongable and the elements of G are distance preserving transformations of* $[R, \varrho]$. *If G is taken with the compact-open topology then it has no small subgroups.*

The following facts are obvious consequences of well-known theorems. For any $x \in R$ the corresponding subgroup of stability $H_x \subset G$ is compact. Since Γ is transitive the elements of G which carry x into $y \in R$ form a subset $\Psi_x(y)$ of G which is a left coset of H_x, and if the left coset space G/H_x is endowed with the quotient topology then the map $\Psi_x: R \to G/H_x$ thus defined is a homeomorphism. Let $\Pi_x: G \to G/H_x$ be the natural projection then $\Pi(G_0)$ is a component of G/H_x. Since R is connected and homeomorphic to G/H_x the identity component G_0 is transitive on R. If $H_{0_x} = H_x \cap G_0$ then since G_0 is a Lie group the left coset space can be endowed with such a differentiable structure that the operation of G_0 on G_0/H_{0_x} by left translations is differentiable. Taking into account the homeomorphism $\Psi_{0_x}: R \to G_0/H_{0_x}$ defined analogously to Ψ_x, the above assertions yield

Theorem 2. *Let* $\Gamma: G \times R \to R$ *be an effective and transitive transformation group and R have a distance function* ϱ *such that* $[R, \varrho]$ *is a finitely compact convex and regular metric space in which the segments are locally and uniquely prolongable and the elements of G are distance preserving transformation of* $[R, \varrho]$. *If G is taken with the compact-open topology then its identity component* G_0 *is a Lie group and R can be endowed with such a differentiable structure that* $\Gamma_0: G_0 \times R \to R$ *the restriction of* Γ *to* $G_0 \times R$ *is a differentiable transformation group.*

For the sake of some of the subsequent and later arguments the main steps in the construction of the differentiable structure of R are summed up here. (For a detailed presentation see [4].) The tangent space $T_\varepsilon H_{0x}$ of H_{0x} at the identity ε is a subspace of $T_\varepsilon G_0$. Let M be a subspace of $T_\varepsilon G_0$ complementary to $T_\varepsilon H_{0x}$. A neighborhood of $O_x \in T_\varepsilon G_0$ is mapped diffeomorphically onto a neighborhood of ε by $\exp_\varepsilon: T_\varepsilon G_0 \varepsilon \to G_0$ and a neighborhood V of O_x in M is mapped homeomorphically onto a neighborhood U of H_{0x} in G_0/H_{0x} by $\Pi_x \circ \exp_\varepsilon: M \to G_0/H_{0x}$. Let $\bar{\varkappa}_x^{-1}$ be the restriction of $\Pi_x \circ \exp_\varepsilon$ to V, since M can be identified with R^m where $m = \dim M$, a coordinate system $\bar{\varkappa}_x: U \to R^m$ of G_0/H_{0x} is obtained. If $\alpha \in G_0$ the left translation $L_\alpha: G_0 \to G_0$ defines a homeomorphism $\bar{L}_\alpha: G_0/H_{0x} \to G_0/H_{0x}$ and $\bar{\varkappa}_x \circ \bar{L}_\alpha$ is a coordinate system on a neighborhood of $\alpha^{-1} H_{0x}$. Thus a differentiable atlas $\{\bar{\varkappa}_x \circ \bar{L}_\alpha : \alpha \in G_0\}$ of G_0/H_{0x} is constructed and this defines a differentiable structure which does not depend on the particular choice of M. For any $z \in R$ the analogously defined differentiable manifold G_0/H_{0z} is diffeomorphic to G_0/H_{0x}. Therefore the homeomorphism $\Psi_{0x}: R \to G_0/H_{0x}$ defines a differentiable structure of R which does not depend on x. The coordinate system $\varkappa_x = \bar{\varkappa}_x \circ \Psi_{0x}^{-1}: U_x \to R^m$ of R will be called a *canonical coordinate system of the first kind at x.*

If $\gamma: R^1 \to G_0$ is a 1-parameter group and $x \in R$ then the differentiable curve $\varphi: R^1 \to R$ defined by $\varphi(\tau) = \gamma(\tau)(x)$, $\tau \in R^1$ is called the orbit of γ starting at x. Let $\varkappa_x: U_x \to R^m$ be a canonical coordinate system of the first kind at x and d the distance function of R^m. If $v \in M$ has length equal to 1 with respect to d and γ is the 1-parameter group defined by $\gamma_*(0) = v$ then the orbit φ of γ starting at x will be called a *fundamental orbit* of the coordinate system $\varkappa_x$. If $z \in U_x$ and $z \neq x$ then there is a unique fundamental orbit φ of $\varkappa_x$ with $\varphi(\tau) = z$ where $\tau = d(\varkappa_x(z), \varkappa_x(x))$.

Let $\varkappa': U' \to R^m$, $\varkappa'': U'' \to R^m$ be coordinate systems of R with $U' \cap U'' \neq \emptyset$ and $\|\alpha_{ij}(u)\|_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ the Jacobian of the map $\varkappa' \circ \varkappa''^{-1}: \varkappa'(U') \cap \varkappa''(U'') \to R^m$ at $\varkappa''(u)$ for $u \in U' \cap U''$. Let $\lambda(\varkappa', \varkappa'')$ be defined by

$$\lambda(\varkappa', \varkappa'') = \sqrt{(2m-1)m} \cdot \sup \{|\alpha_{ij}(u)| : u \in U' \cap U'',\ i, j = 1, \dots, m\}.$$

If $v \in T_u R$ and $(v'^1, \dots, v'^m), (v''^1, \dots, v''^m)$ are its coordinates in the coordinate systems $\varkappa', \varkappa''$ then obviously

$$\left[\sum_{i=1}^{m} (v'^i)^2\right]^{1/2} \leqq \lambda(\varkappa', \varkappa'') \left[\sum_{i=1}^{m} (v''^i)^2\right]^{1/2}.$$

The following lemma will prove useful in later arguments.

Lemma 2.3. *Any $x \in R$ has a compact neighborhood W such that to every $z \in W$ there is a canonical coordinate sytem of the first kind $\varkappa_z: U_z \to R^n$ at z with the following properties:*

1. *$W \subset U_z$ for $z \in W$;*
2. *there is a bound C with $\lambda(\varkappa_z, \varkappa_x) \leqq C$ for $z \in W$;*
3. *if $S(z) \subset T_z R$ is the set of vectors which are tangent to a fundamental orbit of $\varkappa_z$ then $\bigcup \{S(z): z \in W\}$ is a compact subset of TR.*

Proof. Let $M \subset T_\varepsilon G_0$ a subspace complementary to $T_\varepsilon H_{0x}$ be identified with R^m and $\varkappa_x: U_x \to R^m$ the corresponding canonical coordinate system of the first kind at x. If $z \in R$ then $H_{0z} = \alpha H_{0x} \alpha^{-1}$ for any $\alpha \in G_0$ with $z = \alpha(x)$, therefore $T_\varepsilon H_{0z} = L_{\alpha *} R_{\alpha^{-1} *}(T_\varepsilon H_{0x}) = \operatorname{ad} \alpha_*(T_\varepsilon H_{0x})$. This implies the existence of a neighborhood W' of x such that M is complementary to $T_\varepsilon H_{0z}$ for $z \in W'$. Let $\varkappa_z: U_z \to R^m$ be the canonical coordinate system of the first kind at z defined by M for $z \in W'$. If $y \in U_x$ then there is a ξ in the corresponding neighborhood of ε such that $y = \xi(x)$ and $\varkappa_x(y) = \exp_\varepsilon^{-1}(\xi H_{0x} \cap \tilde{M})$ where $\tilde{M} = \exp_\varepsilon(M)$. There is such a neighborhood W'' of x and $\overline{W}$ of ε that $\xi H_{0x} \cap \tilde{M}\alpha$ is a single point and $\exp_\varepsilon^{-1}(\xi H_{0x} \cap \tilde{M}\alpha)$ defines a coordinate system of R on the neighborhood W'' for $\alpha \in \overline{W}$. There is a neighborhood W''' of x such that for $z \in W'''$ there is a unique $\alpha \in \overline{W}$ with $z = \alpha(x)$ and $\alpha = \alpha H_{0x} \cap \tilde{M}$. Then by $\exp_\varepsilon^{-1}((\xi H_{0x} \cap \tilde{M}\alpha)\alpha^{-1})$ a coordinate system of R is defined on W'' for $z \in W'''$. But $\xi H_{0x} = \xi\alpha^{-1} H_{0z} \alpha = \eta H_{0z} \alpha$ with $\eta = \xi\alpha^{-1}$ and $y = \eta(z)$, therefore

$\exp^{-1}((\xi H_{0x}\cap\tilde{M}\alpha)\alpha^{-1})=\exp_\varepsilon^{-1}(\eta H_{0z}\cap\tilde{M})=\varkappa_z(y)$. Let W be a compact neighborhood of x with $W\subset W'\cap W''\cap W'''$. Then $W\subset U_z$ for $z\in W$ and the existence of the bound C follows from the differentiability of the coordinate systems and from the fact that α depends continuously on z, with a possible restriction of U_z to a compact neighborhood $U_z'\subset W$. Since $S(z)$, $z\in W$ is compact $\cup\{S(z):z\in W\}$ is compact as well.

A field of canonical coordinate systems of the first kind $\varkappa_z:U_z\to R^m$, $z\in W$ defined according the preceding proof will be called *normal.*

Let $M\subset \dot{T}_\varepsilon G_0$ be a subspace complementary to $T_\varepsilon H_{0x}$ and $\{w_1,\dots,w_m\}\subset M$ a base of M. Then $v=\sum_{i=1}^{m}\alpha^i v_i$ is unique for $v\in M$ and by $\sigma(v)=\exp(\alpha^1 v_1)\dots\exp(\alpha^m v_m)$ a map $\sigma:M\to G_0$ is defined. With methods similar to those applied at the definition of canonical coordinates of the first kind (see [4]) it can be shown that $\bar{\varkappa}^{-1}=\Pi_x\circ\sigma$ maps diffeomorphically a neighborhood of O_ε in M onto a neighborhood of H_{0x} in G_0/H_{0x}. Thus $\varkappa=\bar{\varkappa}\circ\psi_{0x}:U\to R^m$ is a coordinate system of R which will be called a canonical *coordinate system of the second kind at* x. The proof of the following lemma is obvious.

Lemma 2.4. *Let* $\{v_1,\dots,v_m\}$ *be a base of* T_xR *then there are* 1*-parameter groups* γ_i *of* G_0 *with orbits* φ_i *starting at* x *and a canonical coordinate system of the second kind* $\varkappa:U\to R^m$ *at* x *such that* $\varphi_{i*}(0)=v_i$ *for* $i=1,\dots,m$ *and* $z=\gamma_1(z^1)\circ\dots\circ\gamma_m(z^m)(x)$ *for* $z\in U$ *with* $\varkappa(z)=(z^1,\dots,z^m)$.

3. The introduction of the Riemannian metric

Let $\varphi:R^1\to R$ be the orbit of the 1-parameter group $\gamma:R^1\to G_0$ starting at $x\in R$, then in consequence of the fact that φ is rectifiable $\gamma^*(x)=\lim_{\tau\to 0}\frac{\varrho(\varphi(\tau),\varphi(0))}{|\tau|}$ exists. This defines a function $\gamma^*:R\to R^1$ which is constant on the orbits of γ, and it will be called the *velocity function* of γ. The value φ^* of γ^* on the orbit φ will be called the *velocity of the orbit.* An orbit is constant obviously if and only if its velocity is zero.

Lemma 3.1. *The velocity function* γ^* *of a* 1*-parameter group* γ: $R^1\to G_0$ *is continuous.*

Proof. Let φ be the orbit of γ starting at $x\in R$ and define $f_n:R\to R^1$ for $n=1,2,\dots$ by $f_n(x)=2^n\varrho(\varphi(1/2^n),\varphi(0))$. The functions f_n are continuous, $f_{n+1}(x)\geqq f_n(x)$ and $\gamma^*(x)=\lim_{n\to\infty}f_n(x)$ hold for every $x\in R$. These imply the assertion.

A closer relation of the distance function ϱ and the differentiable structure of R is expressed by

Lemma 3.2. *If $\varkappa: U \to R^m$ is a coordinate system of R at x and d the distance function of R^m then there exist a neighborhood $V \subset U$ of x and a $\delta > 0$ such that $d(\varkappa(a), \varkappa(b)) \geqq \delta \cdot \varrho(a, b)$ if $a, b \in V$.*

Proof. Let $\bar{\varkappa}: U \to R^m$ be a canonical coordinate system of the second kind of R at x. There is a $\tilde{\delta} > 0$ such that $d(\varkappa(a), \varkappa(b)) \geqq \tilde{\delta} d(\bar{\varkappa}(a), \bar{\varkappa}(b))$ for $a, b \in U \cap \bar{U}$. If $\gamma_1, \ldots, \gamma_m$ are the 1-parameter groups which define $\bar{\varkappa}$, then by the preceding lemma there are a neighborhood $V \subset U \cap \bar{U}$ of x and a K such that $\gamma_1^*(z), \ldots, \gamma_m^*(z) \leqq K$ for $z \in V$. If $\bar{\varkappa}(a) = (\alpha^1, \ldots, \alpha^m)$, $\bar{\varkappa}(b) = (\beta^1, \ldots, \beta^m)$ for $a, b \in V$ then

$$\varrho(a, b) = \varrho\big(\gamma_1(\alpha^1) \circ \ldots \circ \gamma_m(\alpha^m)(x), \gamma_1(\beta^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) \leqq$$

$$\leqq \varrho\big(\gamma_1(\alpha^1) \circ \gamma_2(\alpha^2) \circ \ldots \circ \gamma_m(\alpha^m)(x), \gamma_1(\alpha^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) +$$

$$+ \varrho\big(\gamma_1(\alpha^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x), \gamma_1(\beta^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) \leqq$$

$$\leqq \varrho\big(\gamma_2(\alpha^2) \circ \ldots \circ \gamma_m(\alpha^m)(x), \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) + K|\beta^1 - \alpha^1| \leqq \ldots$$

$$\ldots \leqq K \sum_{i=1}^{m} |\beta^i - \alpha^i| \leqq \sqrt{2m} K d(\bar{\varkappa}(a), \bar{\varkappa}(b)).$$

Therefore the assertion of the lemma holds with $\delta = \frac{1}{\sqrt{2m}K} \tilde{\delta}$.

The length of the tangent vectors of a differentiable manifold is usually defined after the introduction of a Riemannian metric. Here the length of tangent vectors of R will be defined at first to be the basic tool in establishing the required Riemannian metric. The velocity of orbits could be naturally considered as the length of their tangent vectors. The following lemma serves to prepare a general definition on this basis.

Lemma 3.3. *Let $\varphi: R^1 \to R$ be an orbit starting at $x \in R$ and $\varphi_*(0) \neq O_*$. If $\psi: [0, \alpha] \to R$ is a curve differentiable at 0 and $\psi(0) = x$, $\psi_*(0) = \lambda \cdot \varphi_*(0)$, $\lambda \geqq 0$, then $\lim_{\tau \to 0} \frac{\varrho(\psi(\tau), \psi(0))}{\tau} = \lambda \varphi^*$.*

Proof. There is a canonical coordinate system of the second kind $\varkappa: U \to R^m$ at x with $\varkappa \circ \varphi(\tau) = (\tau, 0, \ldots, 0)$ for $\varphi(\tau) \in U$ by Lemma 2.4. Therefore $\lim_{\tau \to 0} \frac{d(\varkappa \circ \varphi(\tau), \varkappa(x))}{\varrho(\varphi(\tau), \tau(x)} = \frac{1}{\varphi^*}$ and for a sufficiently small $\bar{\tau} > 0$ there is a $\tau > 0$ such that $\tau = d(\varkappa \circ \varphi(\tau), \varkappa(x)) = d(\varkappa \circ \psi(\bar{\tau}), \varkappa(x))$ and $\tau \to 0$ if $\bar{\tau} \to 0$. Hence

$$\left| \frac{\varrho(\varphi(\tau), x)}{\tau} - \frac{\varrho(\varphi(\tau), \psi(\bar{\tau}))}{\tau} \right| \leqq \frac{\tau(\psi(\bar{\tau}), x)}{\tau} \leqq \frac{\varrho(\varphi(\tau), x)}{\tau} + \frac{\varrho(\varphi(\tau), \psi(\bar{\tau}))}{\tau}.$$

If $\varkappa\circ\psi(\bar\tau)=(\psi^1(\bar\tau),\dots,\psi^m(\bar\tau))$ and $\bar\tau\to+0$ then

$$\limsup\frac{d(\varkappa\circ\varphi(\tau),\varkappa\circ\varphi(\bar\tau))}{\tau}=\limsup\left[2\left(1-\frac{\psi^1(\bar\tau)}{\tau}\right)\right]^{1/2}=$$

$$=\limsup\left[2\left(1-\frac{\psi^1(\bar\tau)}{d(\varkappa\circ\psi(\bar\tau),x(x))}\right]^{1/2}=0.$$

Therefore

$$\limsup\frac{\varrho(\varphi(\tau),\psi(\bar\tau))}{\tau}\leqq$$

$$\leqq\limsup\frac{\varrho(\varphi(\tau),\psi(\bar\tau))}{d(\varkappa\circ\varphi(\tau),\varkappa\circ\psi(\bar\tau))}\cdot\limsup\frac{d(\varkappa\circ\varphi(\tau),\varkappa\circ\psi(\bar\tau))}{\tau}=0$$

in consequence of the preceding lemma. This implies with respect to above inequalities that if $\tau\to+0$ then $\lim\frac{\varrho(\psi(\bar\tau),x)}{\tau}=\varphi^*$. But then $\lim\frac{\varrho(\psi(\bar\tau),x)}{\bar\tau}=$ $=\lim\frac{\varrho(\psi(\bar\tau),x)}{\tau}\cdot\lim\frac{d(\varkappa\circ\psi(\bar\tau),\varkappa(x))}{d(\varkappa\circ\varphi(\bar\tau),\varkappa(x))}=\varphi^*\cdot\lambda.$

Corollary. *If φ,ψ are orbits with $\varphi_*(0)=\psi_*(0)$ then $\varphi^*=\psi^*$.*

On account of the above corollary a function $F:TR\to R^1$ can be defined on the tangent bundle TR of R as follows: let $F(v)$ for a $v\in TR$ be the velocity φ^* of any orbit φ such that $\varphi_*(0)=v$. This function F will be called the *length of tangent vectors*. Obviously $F(v)=0$ if and only if $v=O_*$ for some $x\in R$ and in consequence of the preceding lemma F is positively homogeneous of order 1 on every tangent space of R. In order to show the continuity of F some preliminaries are needed. These are provided by

Lemma 3.4. *Let $\gamma_i:R^1\to G_0$ $(i=0,1,\dots)$ be 1-parameter groups with $\gamma_{0*}(0)=$ $=\lim_{i\to\infty}\gamma_{i*}(0)$. If φ_i is the orbit of γ_i starting at x_i and $x_0=\lim_{i\to\infty}x_i$ then $\varphi_0^*=\lim_{i\to\infty}\varphi_i^*$.*

Proof. It is suitable to consider the special case $x_i=x_0$ $(i=1,2,\dots)$ separately. Let $\varkappa:U\to R^m$ be a coordinate system of R at x_0 with $\varkappa(x_0)=(0,0,\dots,0)$. Since $\gamma_0(\tau)=\lim_{i\to\infty}\gamma_i(\tau)$ for $\tau\in R^1$ and $\Gamma:G_0\times R\to R$ is continuous $\varphi_0(\tau)=\lim_{i\to\infty}\varphi_i(\tau)$ for $\tau\in R^1$. Therefore there is a $\delta>0$ and a N such that $\varphi_i(\tau)\in U$ if $|\tau|<\delta$ and $i=0$ or $i\geqq N$. Since φ_i is differentiable $\varkappa\circ\varphi_i(\tau)=(a_i^1\tau+\tau\varepsilon_i^1(\tau),\dots,a_i^m\tau+\tau\varepsilon_i^m(\tau))$ if $|\tau|\leqq\delta$ and $i=0$ or $i\geqq N$, where $\varepsilon_i^l(\tau)=O(\tau)$, for $l=1,\dots,m$. Let $\eta_i(\tau)$, $i=0,1,\dots$, $\tau\in R^1$ be defined by $\varphi_i^*=\frac{\varrho(\varphi_i(\tau),\varphi_i(0))}{|\tau|}+\eta_i(\tau)$ and $\eta_i(0)=0$. In consequence of lemma 3.2, there is a K with $K\cdot d(\varkappa\circ\varphi_0(\tau),\varkappa\circ\varphi_i(\tau))\geqq\varrho(\varphi_0(\tau),\ \varphi_i(\tau))\geqq$

$\geqq |\tau| \cdot |\varphi_0^* - \varphi_i^* + \eta_i^*(\tau) - \eta_0(\tau)|$ if $|\tau| \leqq \delta$ and $i \geqq N$. Therefore $K \cdot |\tau| \left[\sum_{l=1}^{m} (a_0 - a_i)^2\right]^{1/2} + K \cdot |\tau| \cdot \left[\sum_{l=1}^{m} (\varepsilon_0^l(\tau) - \varepsilon_i^l(\tau))^2\right]^{1/2} \geqq |\tau| |\varphi_0^* - \varphi_i^* + \eta_i(\tau) - \eta_0(\tau)|$ and $\tau \to 0$ yields $K \left[\sum_{l=1}^{m} (a_0^l - a_i^l)^2\right]^{1/2} \geqq |\varphi_0^* - \varphi_i^*|$. But the coordinates of $\varphi_{i*}(0)$ are $(a_i^1, \ldots, a_i^m)$ in the coordinate system $\varkappa$ and $\varphi_{0*}(0) = \lim_{i \to \infty} \varphi_{i*}(0)$ by the continuity of the differential $\Gamma_*: T(G_0 \times R) \to TR$ of Γ. Therefore the assertion of the lemma follows in the special case.

In the general case to any preassigned $\vartheta > 0$ there is a neighborhood V of x_0 with $|\gamma_0^*(x) - \gamma_0^*(x_0)| < \frac{\vartheta}{2}$ for $x \in V$ by Lemma 3. 1. Let $X_0: R \to TR$ be the Killing vector field corresponding to γ_0. In consequence of Lemma 2. 3 there is a normal field of canonical coordinate systems of the first kind $\varkappa_z: U_z \to R^m$ on a compact neighborhood U of x_0. Let $(\alpha_i^1, \ldots, \alpha_i^m)$ respectively $(\bar{\alpha}_i^1, \ldots, \bar{\alpha}_i^m)$ be the coordinates of $\varphi_{i*}(0)$ and $X_i(x_i)$ for $x_i \in U$ in the coordinate system $\varkappa_{x_0}: U_{x_0} \to R^m$. Since $\lim_{i \to \infty} \varphi_{i*}(0) = \varphi_{0*}(0) = X_0(x_0) = \lim_{i \to \infty} X_0(x_i)$, there is a neighborhood $V' \subset U_{x_0}$ of x_0 with $\left[\sum_{l=1}^{m} (\alpha_i^l - \bar{\alpha}_i^l)^2\right]^{1/2} \leqq \left[\sum_{l=1}^{m} (\alpha_i^l - \bar{\alpha}_i^l)^2\right]^{1/2} + \left[\sum_{l=1}^{m} (\alpha_0^l - \bar{\alpha}_0^l)\right]^{1/2} \leqq \frac{\vartheta}{2C^2 K}$, where C is the upper bound given in Lemma 2. 3 and K is an upper bound guaranteed by Lemma 3. 2 for the coordinate system $\varkappa_{x_0}$. Let $(\xi_i^1, \ldots, \xi_i^m)$ respectively $(\bar{\xi}_i^1, \ldots, \bar{\xi}_i^m)$ be the coordinates of $\varphi_{i*}(0)$ and $X_0(x_i)$ in the coordinate system $\varkappa_{x_i}$ and K_{x_i} an upper bound given by Lemma 3. 2 for $\varkappa_{x_i}$ in case of $x_i \in U$. Then

$$|\varphi_i^* - \varphi_0^*| = |\gamma_i^*(x_i) - \gamma_0^*(x_0)| \leqq |\gamma_i^*(x_i) - \gamma_0^*(x_i)| + |\gamma_0^*(x_i) - \gamma_0^*(x_0)| \leqq$$

$$\leqq K_{x_i} \left[\sum_{l=1}^{m} (\xi_i^l - \bar{\xi}_i^l)^2\right]^{1/2} + \frac{\vartheta}{2} \leqq K_{x_i} \cdot \lambda(\varkappa_{x_i}, \varkappa_{x_0}) \left[\sum_{l=1}^{m} (\alpha_i^l - \bar{\alpha}_i^l)^2\right]^{1/2} + \frac{\vartheta}{2} \leqq$$

$$\leqq K \cdot \lambda^2(\varkappa_{x_i}, \varkappa_{x_0}) \frac{\vartheta}{2C^2 K} + \frac{\vartheta}{2} \leqq \vartheta$$

if $x_i \in V \cap V'$.

Lemma 3. 5. *The function $F: TR \to R^1$ is continuous.*

Proof. Let $v_i \in T_{x_i}R$, $i = 0, 1, \ldots$ be such that $v_0 = \lim_{i \to \infty} v_i$. In order to prove $F(v_0) = \lim_{i \to \infty} F(v_i)$ it suffices, on account of the preceding lemma, to show the existence of 1-parameter groups γ_i such that if φ_i is the orbit of γ_i starting at x_i then $v_i = \varphi_{i*}(0)$ and $\gamma_{0*}(0) = \lim_{i \to \infty} \gamma_{i*}(0)$. Let $\psi_{0x_0}: R \to G_0/H_{0x_0}$ be the diffeomorphism defined at the introduction of the differentiable structure of R and $M \subset T_\varepsilon G_0$ a subspace such that $\Pi_{x_0} \circ \exp_\varepsilon: M \to G_0/H_{0x_0}$ is diffeomorphic on a neighborhood V of O_ε in M. Then a neighborhood V' of x_0 exists on which

$\Phi = (\exp_\varepsilon \circ (\Pi_{x_0} \circ \exp_\varepsilon)^{-1} \circ \psi_{0x_0})^{-1} : V' \to G_0$ is diffeomorphic. Put $\bar{x}_i = \Phi(x_i)$, $\bar{v}_i = \Phi_*(v_i)$ and $\tilde{v}_i = R_{\bar{x}_i^{-1}*}(\bar{v}_i)$ for $x_i \in V'$. Let γ_i be the 1-parameter group with $\gamma_{i*}(0) = \tilde{v}_i$ and φ_i the orbit of γ_i starting at x_i for i with $x_i \in V'$. Then $\bar{v}_0 = \lim_{i\to\infty} \bar{v}_i$ by the continuity of Φ_* and $\tilde{v}_0 = \lim_{i\to\infty} R_{\bar{x}_i^{-1}*} \bar{v}_i = \lim_{i\to\infty} \tilde{v}_i = v_0$ by the simultaneous continuity of $R_{\bar{x}*}$ in its argument and in $\bar{x}$. Hence $\gamma_{0*}(0) = \lim_{i\to\infty} \gamma_{i*}(0)$. But $x_i = \bar{x}_i(x_0)$ therefore $\varphi_i(\tau) = \gamma_i(\tau) \cdot \bar{x}_i(x_0)$ for sufficiently small $|\tau|$. Thus $\varphi_i = \Phi^{-1} \circ R_{\bar{x}_i} \circ \gamma_i$ and $\varphi_{i*}(0) = \Phi_*^{-1} \circ R_{\bar{x}_i*}(\gamma_{i*}(0)) = \Phi_*^{-1} R_{\bar{x}_i*}(\tilde{v}_i) = v_i$ if $x_i \in V'$.

What has been proved up to now concerning F can be summarized by stating that the differentiable manifold R with the length of tangent vectors F forms a C^1-Finsler manifold $[R, F]$. The induced metric space of $[R, F]$ can be defined as generally it is done in case of any C^1-Finsler manifold (See [2]) on the following way: If $\psi : [\alpha, \beta] \to R$ is a piecewise C^1-curve of R then $\mathcal{L}_F(\psi) = \int_\alpha^\beta F(\psi_*(\tau))\, d\tau$ is called the *F-length* of ψ. Let $\varrho_F(x, y)$ be the infimum of the F-length of piecewise C^1-curves joining $x, y \in R$, then ϱ_F is a distance function on R. The metric space $[R, \varrho_F]$ is called the *induced metric space* of $[R, F]$. In order to prove $[R, \varrho_F] = [R, \varrho]$ some preliminaries are needed. In what follows these are provided.

If $\psi : [\alpha, \beta] \to R$ is a continuous curve and it is rectifiable in the metric space $[R, \varrho]$ then its length $\mathcal{L}_\varrho(\psi)$ will be called its *ϱ-length*. The following lemma can be proved on essentially the same lines as an other one formulated for the case of symmetric manifolds (see [8]).

Lemma 3.6. *If $\psi : [\alpha, \beta] \to R$ is a piecewise C^1-curve of the differentiable manifold R then it is rectifiable in the metric space $[R, \varrho]$ and $\mathcal{L}_\varrho(\psi) = \mathcal{L}_F(\psi)$.*

Since the metric space $[R, \varrho]$ is finitely compact and convex this lemma has the following obvious consequence:

Lemma 3.7. *If $x, y \in R$ then $\varrho(x, y) \leqq \varrho_F(x, y)$.*

If the continuous curve $\psi : [\alpha, \beta] \to R$ is rectifiable in the metric space $[R, \varrho_F]$ then its length $\mathcal{L}_{\varrho_F}(\psi)$ is called its *ϱ_F-length*. In the case when ψ is a piecewise C^1-curve then evidently $\mathcal{L}_{\varrho_F}(\psi) \leqq \mathcal{L}_F(\psi)$, where according to a result of H. Busemann and W. Mayer (see [1], [2]) the equality holds for any piecewise C_1-curve ψ if and only if F has convex indicatrix in each tangent space $T_x R$ of R. But by Lemma 3.6 and 3.7 $\mathcal{L}_F(\psi) = \mathcal{L}_\varrho(\psi) \leqq \mathcal{L}_{\varrho_F}(\psi)$ for any such curve ψ. These imply

Lemma 3.8. *The function $F : TR \to R^1$ has convex indicatrix in every tangent space of R.*

The proof of the assertion that $\varrho_F(x, y) \leqq \varrho(x, y)$ for $x, y \in R$ requires some technicalities. These are given in the following

Lemma 3.9. *If* $\varkappa: U \to R^m$ *is a coordinate system of* R *at* x *and* d *the distance function of* R^m *then there is a neighborhood* V *of* x *and a* K *such that* $d(\varkappa(a), \varkappa(b)) \leqq K\varrho(a, b)$ *if* $a, b \in V$.

Proof. For the sake of an indirect argument let it be assumed that to any N and in arbitrary neighborhood of x there are points a, b with $d(\varkappa(a), \varkappa(b)) \geqq N\varrho(a, b)$. Let further $\varkappa_z: U_z \to R^m$ be a normal field of canonical coordinate systems on a neighborhood U' of x given according to Lemma 2. 3 and C the corresponding upper bound. Then

$$d(\varkappa(a), \varkappa(b)) \leqq \lambda(\varkappa, \varkappa_x) \cdot \lambda(\varkappa_x, \varkappa_a) d(\varkappa_a(a), \varkappa_a(b)) \leqq \lambda(\varkappa, \varkappa_x) \cdot C d(\varkappa_a(a), \varkappa_a(b))$$

for $a, b \in U \cap U'$. Let $\varphi: R^1 \to R$ be the fundamental orbit of the coordinate system $\varkappa_a$ passing through b and $\varphi(\beta) = b$ then $\dfrac{\varrho(\varphi(0), \varphi(\beta))}{\beta} \leqq \dfrac{C \cdot \lambda(\varkappa, \varkappa_x)}{N}$. Therefore a sequence φ_i, $i = 1, 2, \ldots$ of fundamental orbits of the coordinate systems of the above field can be given with $\lim\limits_{i \to \infty} \dfrac{\varrho(\varphi_i(\beta_i), \varphi_i(0))}{|\beta_i|} = 0$ where $\lim\limits_{i \to \infty} \beta_i = 0$. In consequence of Lemma 2. 3 there is no loss of generality by assuming the existence of a fundamental orbit φ_0 with $\varphi_0(\tau) = \lim\limits_{i \to \infty} \varphi_i(\tau)$, $\tau \in R^1$. Let $\eta_i(\tau)$ be defined by $\varphi_i^* = \dfrac{\varrho(\varphi_i(\tau), \varphi_i(0))}{|\tau|} + \eta_i(\tau)$ and $\eta_i(0) = 0$ for $i = 0, 1, \ldots$ and $\tau \in R^1$. If $\vartheta > 0$ is given then there is such a $\delta > 0$ that $\eta_0(\tau) \leqq \dfrac{\vartheta}{2}$ for $|\tau| \leqq \delta$ and a L with $|\eta_i(\delta) - \eta_0(\delta)| \leqq \dfrac{\vartheta}{2}$ for $i \geqq L$. But obviously $\eta_i(\tau)$ is decreasing for $\tau < 0$ and increasing for $\tau > 0$, therefore $\eta_i(\tau) \leqq \eta_i(\delta) \leqq |\eta_i(\delta) - \eta_0(\delta)| + \eta_0(\delta)$, if $|\tau| \leqq \delta$ and $i \geqq L$. Therefore in consequence of Lemma 2.3 and 3. 4 the equality $\varphi_0^* = \lim\limits_{i \to \infty} \varphi_i^* = 0$ holds in contradiction with the fact that φ_0 is a fundamental orbit.

Lemma 3.10. *If* $x, y \in R$ *then* $\varrho(x, y) \geqq \varrho_F(x, y)$.

Proof. It suffices to prove the inequality for the case when x, y and a metric segment joining them are in the coordinate neighborhood U of a coordinate system $\varkappa: U \to R^m$ and bounds δ, K of Lemma 3. 2 and 3. 9 exist for U. Let $\varphi: [\alpha, \beta] \to R$ be a segment of $[R, \varrho]$ with $\varphi(\alpha) = x$ and $\varphi(\beta) = y$. In consequence of the preceding lemma $\varkappa \circ \varphi: [\alpha, \beta] \to R^m$ is a rectifiable curve of R^m and therefore $F(\varphi_*(\tau)) = 1$ for almost every $\tau \in [\alpha, \beta]$ by Lemma 3. 3. Hence $\varrho(x, y) = \int\limits_\alpha^\beta F(\varphi_*(\tau)) d\tau$. Let a sequence of subdivisions of $[\alpha, \beta]$ be given by $\alpha = \tau_{0,i} < \tau_{1,i} < \cdots < \tau_{n_i - 1, i} < \tau_{n_i, i} = \beta$ $(i = 1, 2, \ldots)$, where the ith subdivision is a refinement of the $(i-1)$th with

$$\lim_{i \to \infty} \max \{\tau_{l,i} - \tau_{l-1,i} : l = 1, \ldots, n_i\} = 0$$

and $\varkappa\circ\varphi$ is differentiable at $\tau_{l,i}$ for $l=1,\dots,n_{i-1}$ $(i=1,2,\dots)$. If i is large enough then the coordinate polygon inscribed in $\varkappa\circ\varphi$ corresponding to the i th subdivision exists, i.e. there is a map $\psi_i:[\alpha,\beta]\to R^m$ where $\psi_i(\tau)=\varkappa\circ\varphi(\tau_{j,i})+$ $+\frac{\tau-\tau_{j,i}}{\tau_{j+1,i}-\tau_{j,i}}(\varkappa\circ\varphi(\tau_{j+1,i})-\varkappa\circ\varphi(\tau_{j,i}))$ for $\tau\in[\tau_{j,i},\tau_{j+1,i}]$, $j=0,1,\dots,n_i-1$. The F-length $\mathscr{L}_F(\psi_i)$ of ψ_i is $\sum_{j=0}^{n_i-1}\int_{\tau_{j,i}}^{\tau_{j+1,i}}F(\psi_{i*}(\tau))d\tau$. But obviously $\varphi_*(\tau)=\lim_{i\to\infty}\psi_{i*}(\tau)$ if $\tau=\tau_{j,i}$ for some i,j and $\alpha<\tau<\beta$, therefore $F(\varphi_*(\tau))=1=\lim_{i\to\infty}F(\psi_{i*}(\tau))$ for such τ by Lemma 3. 5. Let $f_i:[\alpha,\beta]\to R^1$ be defined by $f_i(\tau)=F(\psi_{i*}(\tau))$ for $\tau\in[\tau_{j,i},\tau_{j+1,i}]$, $j=0,1,\dots,n_i-1$ and sufficiently large i. Then $F(\varphi_*(\tau))=\lim_{i\to\infty}f_i(\tau)$ for almost every $\tau\in[\alpha,\beta]$ and the functions f_i are uniformly bounded since

$$F(\psi_{i*}(\tau)) = \lim_{\tau-\tau_{j,i}+0}\frac{\varrho(\psi_i(\tau),\psi_i(\tau_{j,i}))}{\tau-\tau_{j,i}}\leqq$$

$$\leqq \limsup_{\tau\to\tau_{j,i}+0}\frac{\varrho(\psi_i(\tau),\psi_i(\tau_{j,i}))}{d(\varkappa\circ\psi_i(\tau),\varkappa\circ\psi_i(\tau_{j,i}))}\cdot\limsup_{\tau\to\tau_{j,i}+0}\frac{d(\varkappa\circ\psi_i(\tau),\varkappa\circ\psi_i(\tau_{j,i}))}{\tau-\tau_{j,i}}=$$

$$\leqq \limsup_{\tau\to\tau_{j,i}+0}\frac{\varrho(\psi_i(\tau),\psi_i(\tau_{j,i}))}{d(\varkappa\circ\psi_i(\tau),\varkappa\circ\psi_i(\tau_{j,i}))}\cdot\frac{d(\varkappa\circ\psi_i(\tau_{j+1,i}),\varkappa\circ\psi_i(\tau_{j,i}))}{\tau_{j+1,i}-\tau_{j,i}}\leqq\frac{1}{\delta}K,$$

where $\delta>0$ and K are bounds given by Lemma 3. 2 and 3. 9. Therefore by Lebesgue's theorem $\varrho(x,y)=\lim_{i\to\infty}\int_\alpha^\beta f_i(\tau)d\tau$. But if a $\vartheta>0$ is given then

$$\left|\mathscr{L}_F(\psi_i)-\int_\alpha^\beta f_i(\tau)\,d\tau\right|\leqq\sum_{j=0}^{n_i-1}\int_{\tau_{j,i}}^{\tau_{j+1,i}}\left|F(\psi_{i*}(\tau))-F(\psi_{i*}(\tau_{j,i}))\right|d\tau<\vartheta$$

if i is large enough on account of Lemma 3. 5 and of the fact that the $F(\psi_{i*}(\tau_{j,i}))$ are uniformly bounded. Thus $\varrho(x,y)=\lim_{i\to\infty}\mathscr{L}_F(\psi_i)\geqq\varrho_F(x,y)$.

The above lemma and its previous counterpart give

Lemma 3. 11. $[R,\varrho_F]=[R,\varrho]$.

The next step is to show that what F defines on R is actually a Riemannian metric. In proving this the following lemma is essential.

Lemma 3. 12. *If $v_1, v_2\in T_xR$ are linearly independent and $\varphi_1,\varphi_2:R^1\to R$ are orbits starting at x with $\varphi_{i*}(0)=v_i$, $i=1,2$ then* $\lim_{\tau_1,\tau_2\to 0}\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{F(\tau_1v_1-\tau_2v_2)}=1.$

Proof. In fact this is a special case of a result of H. Busemann and W. Mayer in a changed form. To show this let $\varkappa:U\to R^m$ be a canonical coordinate system of the second kind at x with $\varkappa\circ\varphi_1(\tau_1)=(\tau_1,0,0,\dots,0)$ and $\varkappa\circ\varphi_2(\tau_2)=(0,\tau_2,0,\dots,0)$ for $\varphi_1(\tau_1),\varphi_2(\tau_2)\in U$. If $v\in T_zR$, $z\in U$ and $\varkappa(z)=(z^1,\dots,z^m)$, $v=(v^1,\dots,v^m)$ then $F(v)$ is given by $F_\varkappa(z^1,\dots,z^m;v^1,\dots,v^m)$ in the coordinate system $\varkappa$. Let $\psi:[0,1]\to U$

be defined by $\varkappa\circ\psi(\tau)=\varkappa\circ\varphi_2(\tau_2)+\tau(\varkappa\circ\varphi_1(\tau_1)-\varkappa\circ\varphi_2(\tau_2))$ for sufficiently small τ_1,τ_2, then $\psi_*(\tau)=(\tau_1,-\tau_2,0,\dots,0)$. Therefore

$$F(\tau_1v_1-\tau_2v_2)=F(0,\dots,0;\ \tau_1,-\tau_2,0,\dots,0)=$$

$$=\int_0^1 F_\varkappa(0,\dots,0;\ \tau_1,-\tau_2,0,\dots,0)d\tau=M(\psi)$$

which is a quantity introduced by H. BUSEMANN and W. MAYER, and according to their result $\dfrac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{M(\psi)}\to 1$ if $\tau_1,\tau_2\to 0$ (see [2]).

The length of tangent vectors $F:TR\to R^1$ defines a norm in each tangent space of R and the Finsler manifold $[R,F]$ is Riemannian if and only if all these norms are euclidean. Therefore to prove that $[R,F]$ is Riemannian it suffices to show that in the tangent spaces normed by F the metric angle of segments exist (see [7]). In doing this the same methods are used as applied by W. RINOW in analogous questions (see [7]).

Lemma 3.13. *In the tangent spaces T_xR of R normed by F the metric angle of segments exists.*

Proof. Let $v_1,v_2\in T_xR$ be linearly independent with $F(v_1)=F(v_2)=1$ and φ_1,φ_2 orbits starting at x with $\varphi_{i*}(0)=v_i$, $i=1,2$. Then

$$\omega(\tau_1,\tau_2)=\left|\cos\gamma(x;\varphi_1(\tau_1),\varphi_2(\tau_2))-\cos\gamma(0_x;\tau_1v_1,\tau_2v_2)\right|=$$

$$=\left|\frac{\varrho(x,\varphi_1(\tau_1))^2+\varrho(x,\varphi_2(\tau_2))^2-\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2}{2\varrho(x,\varphi_1(\tau_1))\cdot\varrho(x,\varphi_2(\tau_2))}-\frac{\tau_1^2+\tau_2^2-F(\tau_1v_1-\tau_2v_2)^2}{2\tau_1\tau_2}\right|.$$

If $\eta_i(\tau)$, $\tau\in R^1$, $i=1,2$ are the functions introduced in Lemma 3.4 then

$$\omega(\tau_1,\tau_2)=\left|\frac{(\tau_1+\tau_1\eta_1(\tau_1))^2+(\tau_2+\tau_2\eta_2(\tau_2))^2+\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2}{2(\tau_1+\tau_1\eta_1(\tau_1))\cdot(\tau_2+\tau_2\eta_2(\tau_2))}-\right.$$

$$\left.-\frac{\tau_1^2+\tau_2^2-F(\tau_1v_1-\tau_2v_2)^2}{2\tau_1\tau_2}\right|\leqq\left|\frac{\tau_1^2(1+\eta_1(\tau_1))^2+\tau_2^2(1+\eta_2(\tau_2))^2}{2\tau_1\tau_2(1+\eta_1(\tau_1))(1+\eta_2(\tau_2))}-\frac{\tau_1^2+\tau_2^2}{2\tau_1\tau_2}\right|+$$

$$+\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2\left|\frac{1}{2\tau_1\tau_2}-\frac{1}{2\tau_1\tau_2(1+\eta_1(\tau_1))(1+\eta_2(\tau_2))}\right|+$$

$$+\left|\frac{F(\tau_1v_1-\tau_2v_2)^2}{2\tau_1\tau_2}-\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2}{2\tau_1\tau_2}\right|\leqq\frac{1}{2}\left\{\frac{\tau_1}{\tau_2}\left|\frac{1+\eta_1(\tau_1)}{1+\eta_2(\tau_2)}-1\right|+\frac{\tau_2}{\tau_1}\left|\frac{1+\eta_2(\tau_2)}{1+\eta_1(\tau_1)}-1\right|+\right.$$

$$+\left(\frac{\tau_1}{\tau_2}\frac{1+\eta_1(\tau_1)}{1+\eta_2(\tau_2)}+\frac{\tau_2}{\tau_1}\frac{1+\eta_2(\tau_2)}{1+\eta_1(\tau_1)}+2\right)\cdot(\eta_1(\tau_1)+\eta_2(\tau_2)+\eta_1(\tau_1)\eta_2(\tau_2))+$$

$$\left.+\left(\frac{\tau_1}{\tau_2}+\frac{\tau_2}{\tau_1}+2\right)\cdot\left(1+\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{F(\tau_1v_1-\tau_2v_2)}\right)\cdot\left|1-\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{F(\tau_1v_1-\tau_2v_2)}\right|\right\}.$$

Since the orbits have a metric angle $\lim_{\tau_1,\tau_2\to 0}\cos\gamma(x;\ \varphi_1(\tau_1),\ \varphi_2(\tau_2))$ exists. In consequence of the preceding lemma and the above inequalities $\omega(\tau_1,\tau_2)\to 0$ if $\frac{\tau_1}{\tau_2}$ is fixed and $\tau_1,\tau_2\to 0$. The function F as a norm defines a Minkowskian geometry in T_xR therefore the limit of $\gamma(0_x;\ \tau_1 v_1,\tau_2 v_2)$ exists if $\frac{\tau_1}{\tau_2}$ is fixed and $\tau_1,\tau_2\to 0$. These imply that $\lim_{\tau_1,\tau_2\to 0}\gamma(0_x;\ \tau_1 v_1,\tau_2 v_2)$ exists.

What have been proved till now yield that $[R, F]$ is a C^1-Riemannian manifold. With respect to anomalies of such manifolds the following lemma is essential.

Lemma 3.14. *The function $F: TR\to R^1$ defines a C^∞-Riemannian manifold on R.*

Proof. For $\alpha\in G_0$ let $\alpha_*: TR\to TR$ be its differential. If $v\in T_zR$ and $\alpha_*(v)=v'$ then there is an orbit φ starting at z with $\varphi_*(0)=v$. Since $\varphi'=\alpha\circ\varphi$ is a differentiable curve $\varphi'_*(0)=\alpha_*(\varphi_*(0))=v'$. But then $F(v)=F(v')$ in consequence of the fact that α is a distance preserving transformation of $[R, \varrho]$ and of Lemma 3. 3. Therefore α is an isometric transformation of $[R, F]$. Let $v_1, \dots, v_m\in T_xR$ be an orthonormal system and $\varkappa: U\to R^m$ a canonical coordinate system of the second kind at x defined by orbits $\varphi_1, \dots, \varphi_m$ with $\varphi_{i*}(0)=v_i$, $i=1, \dots, m$ according to Lemma 2. 4. Therefore if $z\in U$ and $\varkappa(z)=(z^1, \dots, z^m)$ then $z=\gamma_1(z^1)\circ\cdots\circ\gamma_m(z^m)(x)$ where γ_i is the 1-parameter group which defines φ_i, $i=1, \dots, m$. Let $g_{ij}(z^1, \dots, z^m)$, $i, j=1, \dots, m$ be the components of the Riemannian tensor defined by F with respect to the coordinate system $\varkappa$ for $z\in U$. But $\gamma_{i*}(0)$, $i=1, \dots, m$ are linearly independent therefore 1-parameter groups $\gamma_{m+1}, \dots, \gamma_n$ exist which define a canonical coordinate system of the second kind $\bar{\varkappa}:\bar{U}\to R^n$ of G_0 at ε. Thus $z'^i=\Gamma^i(\alpha^1, \dots, \alpha^n;\ z^1, \dots, z^m)$, $i=1, \dots, m$ if $\alpha\in\bar{U}$, $\bar{\varkappa}(\alpha)=(\alpha^1, \dots, \alpha^n)$, $z\in U$, $\alpha(z)=z'\in U$. The functions Γ^i are C^∞ since $\Gamma: G_0\times R\to R$ is a C^∞-map. In consequence of the special choice of the coordinate systems $u^i=\Gamma^i(u^1, \dots, u^m, 0, \dots, 0;\ 0, \dots, 0)$, $i=1, \dots, m$ for $u\in U$. Since the elements of G_0 are isometric transformations

$$g_{ij}(0, \dots, 0)=\delta_{ij}=$$

$$=\sum_{k,l=1}^{m} g_{kl}(u^1, \dots, u^m)\frac{\partial\Gamma^k(u^1,\dots,u^m,0,\dots,0;0,\dots,0)}{\partial z^i}\,\frac{\partial\Gamma_l(u^1,\dots,u^m,0,\dots,0;0,\dots,0)}{\partial z^j}$$

for $i, j=1, \dots, m$, which considered as a system of equations for the $g_{kl}(u^1, \dots, u^m)$, $k, l=1, \dots, m$ must have a unique solution. This together with the fact that the Γ^i are C^∞-functions yield that the g_{kl} are C^∞ as well, what obviously implies the assertion of the lemma.

It is to be noted that contrary to the circumstance that Lemmas 3. 1-12 do not assume the existence of the metric angle of orbits for the last one this is essential.

In fact Lemma 3. 14 cannot have an analogue in the case of Finsler manifolds as obvious examples of Minkowskian geometries show.

Results of this section are summed up in

Theorem 3. *Let $\Gamma : G_0 \times R \to R$ be a differentiable transformation group and the differentiable manifold R have a distance function ϱ such that the metric space $[R, \varrho]$ is finitely compact and convex. If the elements of G_0 are distance preserving transformations of $[R, \varrho]$ and the orbits of the 1-parameter groups of G_0 are rectifiable and have metric angle in $[R, \varrho]$ then there is a unique Riemannian metric on R such that its induced metric space is $[R, \varrho]$.*

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