The infimum of two projections

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1. The purpose of this note is to give a closed formula for the infimum of two projections in Hilbert space. Heretofore, this infimum has been expressed by an iterative scheme as follows:[1]) If P and Q are projections on closed subspaces M and N, then the projection $P \wedge Q$ on $M \cap N$ is given by

(1) $$P \wedge Q = \lim_{n \to \infty} P(QP)^n.$$

We shall show that

(2) $$P \wedge Q = 2P(P+Q)^+ Q$$

if $M+N$ is closed, where $(P+Q)^+$ denotes the inverse of $P+Q$ restricted to its range. This formula is exactly that given by ANDERSON and DUFFIN [2] for matrices, and the result of this note is that it generalizes to infinite dimensions as stated. We also give an apparently heretofore unstated necessary and sufficient condition for $M+N$ to be closed (namely, that M and N make a positive angle modulo $M \cap N$).

The combination $A(A+B)^+B$, introduced in [2], is called the parallel sum, so named because of its origin in and application to electrical network theory ($(r_1^{-1}+r_2^{-1})^{-1} = r_1(r_1+r_2)^{-1} r_2$ is the resistance arising from resistors r_1 and r_2 in parallel). A second formulation of the parallel sum in finite dimensions is given by ANDERSON [1].

We shall use the following notations: P_M for the projection on M, $R(A)$ and $N(A)$ for range and nullity, M^- and $M^\perp$ for closure and orthocomplement, $A|M$ for restriction.

2. We begin by noting some facts about non-negative bounded operators.

For hermitian operators generally one knows that $R(A) = A\overline{R(A)}$, that A is one-one on $\overline{R(A)}$, and A is invertible on $R(A)$ if and only if $R(A) = \overline{R(A)}$ (closed graph theorem).

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[1]) See [4] for background.

Hence for non-negative operators we have

(i) $R(A)$ closed if and only if A is bounded away from 0 on $R(A)$.

It follows that $R(A)$ is closed if and only if $R(A^{\frac{1}{2}})$ is closed. Since $N(A)=N(A^{\frac{1}{2}})$ (because by positivity $(Ax, x)=0$ if and only if $x=0$), and therefore $\overline{R(A)}=\overline{R(A^{\frac{1}{2}})}$, we have

(ii) $R(A)$ closed implies $R(A)=R(A^{\frac{1}{2}})$.

T. Crimmins has shown that $R(S)+R(T) = (SS^*+TT^*)^{\frac{1}{2}}$ for any bounded operators S and T.[2]) From this we get

(iii) If $R(A)$, $R(B)$, and $R(A+B)$ are closed, then

$$R(A)+R(B)=R(A+B), \tag{3}$$

for $R(A)+R(B) = R(A^{\frac{1}{2}})+R(B^{\frac{1}{2}}) = R((A+B)^{\frac{1}{2}}) = R(A+B)$.

As it follows from the preceding computation that $R(A)+R(B)$ is closed if and only if $R((A+B)^{\frac{1}{2}})$ is closed, we have

(iv) If $R(A)$ and $R(B)$ are closed, a necessary and sufficient condition that $R(A+B)$ be closed is that $R(A)+R(B)$ be closed.

Next, we introduce a generalized inverse (see [3] for background) and the parallel sum. If A has closed range, define A^+ by

$$A^+|R = (A|R)^{-1}, \quad A^+|N = 0. \tag{4}$$

If $A+B$ has closed range, define the parallel sum $A:B$ by

$$A:B = A(A+B)^+ B. \tag{5}$$

This operation satisfies

$$A:B = B:A = (A:B)^*, \quad R(A:B) = R(A)\cap R(B) \tag{6}$$

if $R(A)$ and $R(B)$ are closed, as we shall presently show. The equations $R(A+B) = R(A)+R(B)$ and $R(A:B) = R(A)\cap R(B)$, valid when $R(A)+R(B)$ is closed, show that the set of $A\geqq 0$ with closed range is almost a lattice, in the sense that the sup and inf, inherited from the lattice of subspaces, exist whenever the sum of the ranges is closed. If we now think of projections P and Q in this context and ask what significance attaches to their sup and inf, we are led to the following theorem:

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[2]) The result is unpublished and was told to us by J. P. Williams. Proof: Write $A = \begin{pmatrix} S & -T \\ O & O \end{pmatrix}$ on $H\oplus H$. Then $\{R(S)+R(T)\}\oplus\{0\} = R(A) = R((AA^*))^{\frac{1}{2}} = R((SS^*+TT^*))^{\frac{1}{2}}\oplus\{0\}$.

Theorem. *If P and Q are projections such that $R(P)+R(Q)$ is closed, then* $2P:Q = P\wedge Q$.

Note that by (iv) the hypothesis is equivalent to assuming $R(P+Q)$ closed.

To prove the theorem, we first prove the identities (6) (following [2]). By (iii) we have $R(A), R(B) \subset R(A+B)$. Hence $B(A+B)^+(A+B) = B, (A+B)^+(A+B)A = A$ (because $(A+B)^+(A+B)$ is the projection on $R(A+B)$). Then $A:B = A(A+B)^+B = (A+B-B)(A+B)^+(A+B-A)$ easily reduces to $B(A+B)^+A = B:A$. Since $(A:B)^* = B:A$, we have $(A:B)^* = A:B$. By the commutativity again we have $R(A:B) \subset R(A)\cap R(B)$. If $x\in R(A)\cap R(B)$, then

$$(A:B)(A^+ + B^+)x = A(A+B)^+BB^+x + B(A+B)^+AA^+x = $$
$$= \big(A(A+B)^+ + B(A+B)^+\big)x = P_{R(A+B)}x = x.$$

Hence $R(A:B) = R(A)\cap R(B)$. This last computation simplifies, if A and B are projections P and Q, to $2P(P+Q)^+Qx = 2(P:Q)x = x$ for $x\in R(P)\cap R(Q)$. Thus $2(P:Q)$ is a hermitian idempotent with range $R(P)\cap R(Q)$, and this proves the theorem.

3. We conclude by showing that $R(P)+R(Q)$ is closed if and only if $R(P)$ and $R(Q)$ make a positive angle modulo $R(P)\cap R(Q)$. To establish this, we first consider the case when the two subspaces, which we denote by M and N, are disjoint. Then the assertion is that $M+N$ is closed if and only if

$$|(x, y)| \leqq \|x\|\,\|y\|\,(1-\delta) \quad (\delta > 0) \tag{7}$$

for all $x\in M$, $y\in N$.

Suppose (7) holds, and consider a Cauchy sequence $u_n = x_n + y_n$ in $M+N$. Write $u_{nm} \equiv u_n - u_m$, etc. Then

$$\|u_{nm}\|^2 = \|x_{nm}\|^2 + \|y_{nm}\|^2 + 2\mathrm{Re}\,(x_{nm}, y_{nm}) \geqq \|x_{nm}\|^2 + \|y_{nm}\|^2 -$$
$$-2(1-\delta)\|x_{nm}\|\,\|y_{nm}\| = \{\|x_{nm}\| - \|y_{nm}\|\}^2 + 2\delta\|x_{nm}\|\,\|y_{nm}\|.$$

It follows that $u_{nm}\to 0$ implies $x_{nm}\to 0$, $y_{nm}\to 0$, and therefore that u_n converges in $M+N$.[3])

Conversely, suppose $M+N$ is closed. It then follows that the "coordinate" map $T\colon M+N \to M$ given by $T(x+y) = x$ is continuous. The proof consists in verifying that T has closed graph and then applying the closed graph theorem. This argument is due to KOBER [5], and as the verification is simple we omit it. Since T is bounded, we have

$$\|x\| \leqq A\,\|x\pm y\|$$

[3]) This formulation of the argument is due to B. GLICKFELD.

for all $x \in M$, $y \in N$. Taking $\|x\| = \|y\| = 1$ and squaring, we get

$$1 \leqq A^2(2 \pm 2\,\mathrm{Re}\,(x, y))$$

whence $|\mathrm{Re}\,(x, y)| \leqq 1 - \frac{1}{2A^2}$, and (7) follows from this immediately.

If $M \cap N = I \neq \{0\}$, we pass to the quotient space H/I. If $M+N$ is closed, so is $(M+N)/I = M/I + N/I$ and so M and N make a positive angle modulo I. Conversely, if M/I and N/I make a positive angle H/I, then $M/I + N/I$ is closed, whence so is $M+N$.

References

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(Received January 15, 1971)