Operators with a norm condition

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1. Introduction

A bounded linear operator T on a Hilbert space is called, according to [1], *paranormal* if

(1) $$\|T^2x\| \cdot \|x\| \geqq \|Tx\|^2.$$

Paranormal operators are abundant: a *hyponormal* operator, i.e. $T^*T \geqq TT^*$, is paranormal, and every power of a paranormal operator is again paranormal. We show further that an invertible operator T is paranormal if $\log(T^*T) \geqq \log(TT^*)$.

Two bounded linear operators T and S *double commute*, by definition, if T commutes with both S and S^*. The sum and product of two double commuting hyponormal operators are hyponormal. The corresponding assertion is shown not to hold for paranormal operators. We prove, however, that the product of two double commuting operators, one of which is paranormal and the other is hyponormal, is paranormal.

Our central result is that a bounded linear operator T is normal if (and only if) both T and T^* are paranormal and if they have the common kernel. Finally we prove that a paranormal operator is normal if some of its powers is normal.

2. Paranormality

Throughout the paper, $\mathfrak{D}(T)$, $\mathfrak{R}(T)$ and $\mathfrak{N}(T)$ for a (bounded or unbounded) linear operator T denote its domain, range and kernel respectively. The *compression* of T to a closed subspace $\mathfrak{M}$ is the operator PT considered on $\mathfrak{M}$, where P is the projection on $\mathfrak{M}$.

For many purposes, paranormality is more conveniently handled when it is defined by an additive inequality.

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Theorem 1. *A bounded linear operator T is paranormal if and only if*

$$T^{*2}T^2-2\lambda T^*T+\lambda^2 \geqq 0 \qquad (\lambda>0). \tag{2}$$

Proof. Equivalence of (1) and (2) is an immediate consequence of the relation:

$$\|T^2x\|\cdot\|x\| = \inf_{\lambda>0}\tfrac{1}{2}\{\lambda^{-1}\|T^2x\|^2+\lambda\|x\|^2\}.$$

Since $\mathfrak{R}(T^*)+\mathfrak{N}(T)$ is dense, (1) is clearly equivalent to

$$\|T^2T^*x\|\cdot\|T^*x\| \geqq \|TT^*x\|^2, \tag{1'}$$

so that (2) is equivalent to

$$AB^2A-2\lambda A^2+\lambda^2 \geqq 0 \qquad (\lambda>0), \tag{2'}$$

where $A=(TT^*)^{\frac{1}{2}}$ and $B=(T^*T)^{\frac{1}{2}}$.

In order to generalize the result that a hyponormal operator is paranormal, let us recall some definitions. For semi-bounded (from below) selfadjoint operators A and B, the order relation $A\geqq B$ means that

$$\int_{-\infty}^{\infty} t\cdot d(E(t)x,x) \geqq \int_{-\infty}^{\infty} t\cdot d(F(t)x,x),$$

where $\{E(t)\}$ and $\{F(t)\}$ are the resolutions of identity for A and B respectively. When A and B are semi-bounded from above, $A\geqq B$ means, by definition, $-A\leqq -B$. If both A and B are *positive*, i.e. A, $B\geqq 0$, then $A\geqq B$ is equivalent to that $\mathfrak{D}(A^{\frac{1}{2}})\subset\mathfrak{D}(B^{\frac{1}{2}})$ and

$$\|A^{\frac{1}{2}}x\|\geqq\|B^{\frac{1}{2}}x\| \qquad (x\in\mathfrak{D}(A^{\frac{1}{2}})).$$

The spectral theory shows that if $A\geqq B\geqq 0$ and if B has inverse, then A has inverse and $0\leqq A^{-1}\leqq B^{-1}$.

Theorem 2. *A bounded linear operator T is paranormal if* $\mathfrak{N}(T)\subset\mathfrak{N}(T^*)$ *and* $\log(A)\geqq\log(B)$ *where A and B are respectively the compressions of* T^*T *and* TT^* *to* $\overline{\mathfrak{R}(T)}$.

Proof. We may assume, without loss of generality $\|T\|\leqq 1$. $\mathfrak{N}(T)\subset\mathfrak{N}(T^*)$ implies that both A and B have inverse. Take $x\in\mathfrak{D}(B^{-1})$. Then the function

$$\Phi(t):=(A^tx,x)(B^{-t}x,x)-(x,x)^2 \qquad (1\geqq t\geqq 0)$$

is convex. Since $t^{-1}(\alpha^t-1)$ converges monotonously to $\log(\alpha)$ for $\alpha>0$, the spectral theory shows that

$$\Phi'(0)=-\left\|(-\log(A))^{\frac{1}{2}}\right\|^2\cdot\|x\|^2+\left\|(-\log(B))^{\frac{1}{2}}x\right\|^2\cdot\|x\|^2 \geqq 0.$$

With $\Phi(0)=0$ and $\Phi'(0)\geqq 0$ the convexity yields

$$\Phi(1) = (Ax, x)(B^{-1}x, x)-(x, x)^2 \geqq 0,$$

so that

$$(BABy, y)(By, y) \geqq (B^2y, y)^2 \qquad (y\in\mathfrak{R}(T)).$$

Then (1') results, because

$$PBABP=TT^{*2}T^2T^*, \quad PB^2P=(TT^*)^2 \quad \text{and} \quad PBP=TT^*$$

where P is the projection on $\overline{\mathfrak{R}(T)}$. This completes the proof.

If $(T^*T)^s\geqq(TT^*)^s$ for some $s>0$, the condition of Theorem 2 is fulfilled. In fact, $\mathfrak{R}(T)\subset\mathfrak{R}(T^*)$ is trivial. The LOEWNER—HEINZ—KATO theorem (cf. [3] V, § 4) guarantees $(T^*T)^t\geqq(TT^*)^t$ for $0\leqq t\leqq s$, hence $\log(A)\geqq\log(B)$ as in the proof of Theorem 2.

3. Sum and product

The sum and product of two double commuting hyponormal operators are easily shown to be hyponormal. We shall show that the corresponding assertion does not hold for paranormal operators.

If a bounded linear operator T is paranormal, the *tensor product* $T\otimes 1$ (and $1\otimes T$) is paranormal. In fact, for $\lambda>0$

$$(T\otimes 1)^{*2}(T\otimes 1)^2-2\lambda(T\otimes 1)^*(T\otimes 1)+\lambda^2(1\otimes 1) = (T^{*2}T^2-2\lambda T^*T+\lambda^2)\otimes 1 \geqq 0,$$

because the tensor product of two positive operators is positive. We prove, however, that the tensor product $T\otimes T$ is not necessarily paranormal. Then this will show that the product of two double commuting paranormal operators is not necessarily paranormal, because

$$T\otimes T = (T\otimes 1)(1\otimes T)$$

and $T\otimes 1$ double commutes with $1\otimes T$. The construction of such an operator will be based on the idea of P. R. HALMOS (cf. [1]).

When $\mathfrak{H}$ is a Hilbert space, $\mathbf{H}$ denotes the infinite direct sum of copies of H, i.e. $\mathbf{H}=\mathfrak{H}\oplus\mathfrak{H}\oplus\cdots$. $\mathfrak{H}$ itself is identified with the first summand. Given two bounded positive operators A and B on $\mathfrak{H}$, $\mathbf{T}_{A,B,n}$ is the operator on $\mathbf{H}$, which assigns to a vector $\mathbf{x}=\langle x_1, x_2, \ldots\rangle$ the vector $\mathbf{y}=\langle y_1, y_2, \ldots\rangle$ such that $y_1=0$, $y_j=Ax_{j-1}$ $(1<j\leqq n+1)$ and $y_j=Bx_{j-1}$ $(j>n+1)$. Computation shows that the operator $\mathbf{T}_{A,B,n}$ is paranormal if and only if

$$AB^2A-2\lambda A^2+\lambda^2 \geqq 0 \qquad (\lambda>0) \tag{3}$$

and that it is hyponormal if and only if $B^2\geqq A^2$.

Let now $\dim(\mathfrak{H})=2$. Then each linear operator on $\mathfrak{H}$ is represented by the corresponding matrix with respect to a fixed orthonormal basis. Consider the operators

$$C=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} \quad \text{and} \quad D=\begin{pmatrix}1 & 2\\ 2 & 8\end{pmatrix}.$$

Then both C and D are positive and for every $\lambda>0$

$$D-2\lambda C+\lambda^2=\begin{pmatrix}(1-\lambda)^2 & 2(1-\lambda)\\ 2(1-\lambda) & (2-\lambda)^2+4\end{pmatrix}\geqq 0.$$

Observe the operator $T=\mathbf{T}_{A,B,1}$ with $A=C^{\frac{1}{2}}$ and $B=(C^{-\frac{1}{2}}DC^{-\frac{1}{2}})^{\frac{1}{2}}$. Then T is paranormal by (3), but the tensor product $T\otimes T$ is not paranormal. In fact, otherwise by (2)

$$(T\otimes T)^{*2}(T\otimes T)^2-2(T\otimes T)^*(T\otimes T)+1\otimes 1\geqq 0,$$

so that the compression of the left side to the canonical imbedding of $\mathfrak{H}\otimes\mathfrak{H}$ in $\mathbf{H}\otimes\mathbf{H}$ is also positive. However the compression coincides with

$$D\otimes D-2C\otimes C+1\otimes 1=\begin{pmatrix}0 & 0 & 0 & 2\\ 0 & 5 & 2 & 12\\ 0 & 2 & 5 & 12\\ 2 & 12 & 12 & 57\end{pmatrix},$$

which is not positive.

Theorem 3. *Let T and S be double commuting paranormal operators. Then the product TS is paranormal if there are a selfadjoint operator A and bounded positive Borel functions $f(t)$ and $g(t)$ such that*

$$(f(t)-f(s))(g(t)-g(s))\geqq 0 \qquad (-\infty<s,t<\infty)$$

and one of the following holds:

(a) $f(A)=T^*T$ *and* $g(A)=S^*S$,

(b) $f(A)=T^{*2}T^2$ *and* $g(A)=S^*S$,

(c) $f(A)=T^{*2}T^2$ *and* $g(A)=S^{*2}S^2$.

Proof. Remark, first of all, that the assumption implies

$$(f(A)g(A)x,x)\cdot(x,x)\geqq(f(A)x,x)\cdot(g(A)x,x). \tag{4}$$

In fact, let $\{E(t)\}$ be the resolution of identity for A. Then

$$(f(A)g(A)x, x)(x, x)-(f(A)x, x)(g(A)x, x) =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{f(t)g(t)-f(t)g(s)\}d(E(t)x, x)d(E(s)x, x) =$$

$$= \iint_{t>s}(f(t)-f(s))(g(t)-g(s))d(E(t)x, x)d(E(s)x, x) \geqq 0.$$

Double commutativity and (4), when applied to (a), (b) and (c), yield respectively

(a′) $\|TSx\|\cdot\|x\| \geqq \|Tx\|\cdot\|Sx\|$,

(b′) $\|T^2Sx\|\cdot\|x\| \geqq \|T^2x\|\cdot\|Sx\|$,

(c′) $\|T^2S^2x\|\cdot\|x\| \geqq \|T^2x\|\cdot\|S^2x\|$.

Let (a′) hold. Then since both T and S are paranormal,

$$\|T^2S^2x\|\cdot\|S^2x\|\cdot\|Sx\|^2\cdot\|x\| \geqq \|TS^2x\|^2\cdot\|Sx\|^2\cdot\|x\| \geqq$$

$$\geqq \|TSx\|^2\cdot\|S^2x\|^2\cdot\|x\| \geqq \|TSx\|^2\cdot\|S^2x\|\cdot\|Sx\|^2,$$

hence TS is paranormal.

Let (b′) hold. Then since T commutes with S,

$$\|T^2S^2x\|\cdot\|T^2x\|\cdot\|x\| \geqq \|ST^2x\|^2\cdot\|x\| \geqq \|T^2Sx\|\cdot\|T^2x\|\cdot\|Sx\| \geqq \|TSx\|^2\cdot\|T^2x\|,$$

hence TS is paranormal.

Finally let (c′) hold. Since T and S are double commuting paranormal operators, for every $\lambda, \varrho>0$

$$(TS)^{*2}(TS)^2+\lambda^2\varrho^2+\lambda^2S^{*2}S^2+\varrho^2T^{*2}T^2 =$$

$$= (S^{*2}S^2+\varrho^2)(T^{*2}T^2+\lambda^2) \geqq 4\lambda\varrho(TS)^*(TS),$$

so that

$$\|T^2S^2x\|^2+\lambda^2\varrho^2\|x\|^2+\lambda^2\|S^2x\|^2+\varrho^2\|T^2x\|^2 \geqq 4\lambda\varrho\|TSx\|^2.$$

It follows with $\lambda\varrho = \|x\|^{-1}\cdot\|T^2S^2x\|$ and $\lambda^{-1}\varrho = \|T^2x\|^{-1}\cdot\|S^2x\|$ that

$$\|T^2S^2x\|\cdot\|x\|+\|T^2x\|\cdot\|S^2x\| \geqq 2\|TSx\|^2.$$

Now the paranormality of TS results from (c′). This completes the proof.

Theorem 4. *If a paranormal operator T double commutes with a hyponormal operator S, then the product TS is paranormal.*

Proof. Let $\{E(t)\}$ be the resolution of identity for S^*S. By assumption both T^*T and $T^{*2}T^2$ commute with every $E(t)$. Since $S^*S \geqq SS^*$, it follows that for $\lambda > 0$

$$(TS)^{*2}(TS)^2 - 2\lambda(TS)^*(TS) + \lambda^2 \geqq (T^{*2}T^2)(S^*S)^2 - 2\lambda(T^*T)(S^*S) + \lambda^2 =$$

$$= \int_0^\infty (t^2 T^{*2}T^2 - 2\lambda t T^*T + \lambda^2)\, dE(t) \geqq 0,$$

hence TS is paranormal by (2).

Even if both the operators in Theorem 4 are hyponormal, double commutativity can not be replaced by commutativity. To see this, let us construct a hyponormal operator T such that $T^2 - \xi$ is not paranormal for some complex ξ. Then $T - \xi^{\frac{1}{2}}$ and $T + \xi^{\frac{1}{2}}$ are commuting hyponormal operators with non-paranormal product. Since T^2 is paranormal in this case, this example will show that the sum of a paranormal operator and a scalar is not necessarily paranormal.

First we prove that if $S - \xi$ is paranormal for every complex ξ, then

$$\|Sx\|^2 \geqq |(S^2x, x)|. \tag{5}$$

In fact, the assumption implies by (2) that for every $0 \leqq \theta < 2\pi$ and $r > 0$

$$(S^* - re^{-i\theta})^2(S - re^{i\theta})^2 - 2r^2(S^* - re^{-i\theta})(S - re^{i\theta}) + r^4 \geqq 0,$$

so that as $r \to \infty$

$$e^{-2i\theta}S^2 + e^{2i\theta}S^{*2} + 2S^*S \geqq 0.$$

Then (5) result from the arbitrariness of θ.

Let now

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then $D \geqq C \geqq 0$, but

$$26D^2 - 25C^2 = \begin{pmatrix} 105 & 130 \\ 130 & 160 \end{pmatrix}$$

is not positive, hence $4^{\frac{1}{n}} D^2 - C^2$ is not positive, where n is a positive integer so large that $4 \times 25^n < 26^n$. Observe the operators A and B on 2^n dimensional space $\mathfrak{H}$:

$$A = (C \otimes \cdots \otimes C)^{\frac{1}{2}} \quad \text{and} \quad B = (D \otimes \cdots \otimes D)^{\frac{1}{2}}.$$

Since $D \geqq C$ is equivalent to $B^2 \geqq A^2$, the operator $T = \mathbf{T}_{A,B,4}$ on $\mathbf{H}$ is hyponormal, but $4B^4 - A^4$ is not positive, for otherwise $4^{\frac{1}{n}} D^2 \geqq C^2$. Suppose that $T^2 - \xi$ is paranormal for every complex ξ. Then since for $y, z \in \mathfrak{H}$

$$\|T^2\mathbf{x}\|^2 = \|A^2y\|^2 + \|B^2z\|^2 \quad \text{and} \quad (T^4\mathbf{x}, \mathbf{x}) = (A^4y, z)$$

where $x_1=y$, $x_5=z$ and $x_k=0$ for other k, it follows from (5) that

$$\|A^2y\|^2+\|B^2z\|^2 \geqq |(A^4y, z)|.$$

Since y and z are arbitrary, this leads to

$$2\|A^2y\|\cdot\|B^2z\| \geqq |(A^4y, z)|$$

and finally to $4B^4\geqq A^4$, a contradiction.

4. Normality

Paranormality, when combined with other conditions, leads often to normality. For instance, it is known (cf. [2]) that a compact paranormal operator is normal. It is quite trivial that a hyponormal operator with hyponormal adjoint is normal. However, the generalization to paranormal case is not at all trivial. Our proof is based on the following lemma on positive operators.

Lemma. *If bounded positive operators A and B satisfy*

(6) $$2\lambda A^2(A^2+\lambda^2)^{-1} \leqq B \leqq (2\lambda)^{-1}(A^2+\lambda^2) \qquad (\lambda > 0),$$

then they coincide with each other.

Proof. It suffices to prove that B commutes with the resolution of identity $\{E(t)\}$ for A, because under the commutativity assumption (6) and the relation

$$\sup_{\lambda>0} 2\lambda\xi^2(\xi^2+\lambda^2)^{-1} = \xi = \inf_{\lambda>0}(2\lambda)^{-1}(\xi^2+\lambda^2) \qquad (\xi\geqq 0)$$

yield $A=B$ by standard arguments in spectral theory. Since (6) implies $\mathfrak{N}(A)=\mathfrak{N}(B)$, it remains to show that

$$\big(1-E(t+s)\big)B\big(E(t)-E(s)\big) = 0 \qquad (t\geqq s>0).$$

Take x and y such that

$$x = \big(E(t)-E(s)\big)x \quad\text{and}\quad y=\big(1-E(t+s)\big)y.$$

Consider a partition:

$$s=t_0<t_1<\dots<t_n=t \quad\text{with}\quad t_j-t_{j-1}<n^{-1}t,$$

and let

$$x_j=\big(E(t_j)-E(t_{j-1})\big)x \quad\text{and}\quad A_j=2t_j(A^2+t_j^2)^{-1}.$$

Then it follows from (6) that

$$0\leqq B-A^2A_j\leqq A_j^{-1}-A^2A_j =$$
$$=(2t_j)^{-1}(A-t_j)^2(A+t_j)^2(A^2+t_j^2)^{-1}\leqq s^{-1}(A-t_j)^2.$$

Since by definition

$$(A^2 A_j x_j, y) = 0 \quad \text{and} \quad \|(A-t_j)x_j\| \leqq n^{-1} t \|x_j\|,$$

the Schwartz inequality shows that

$$|(Bx_j, y)|^2 = |((B-A^2A_j)x_j, y)|^2 \leqq ((B-A^2A_j)x_j, x_j)\cdot((B-A^2A_j)y, y) \leqq$$
$$\leqq s^{-1}\|B\|\cdot\|(A-t_j)x_j\|^2\cdot\|y\|^2 \leqq (n^2s)^{-1}t^2\|B\|\cdot\|x_j\|^2\cdot\|y\|^2,$$

so that

$$|(Bx, y)|^2 = \left|\sum_{j=1}^{n}(Bx_j, y)\right|^2 \leqq n\cdot\sum_{j=1}^{n}|(Bx_j, y)|^2 \leqq (ns)^{-1}t^2\|B\|\cdot\|x\|^2\|y\|^2.$$

Now $(Bx, y)=0$ results as $n\to\infty$. This completes the proof.

Theorem 5. *A bounded linear operator T is normal if both T and T^* are paranormal and if $\mathfrak{N}(T)=\mathfrak{N}(T^*)$.*

Proof. Let $A=(T^*T)^{\frac{1}{2}}$ and $B=(TT^*)^{\frac{1}{2}}$. Then we have to prove $A^2=B^2$. Since $\mathfrak{N}(A)=\mathfrak{N}(B)$ by assumption, we may assume without loss of generality that both A and B have inverse. Paranormality in assumption means by (2′) that

$$AB^2A-2\lambda A^2+\lambda^2 \geqq 0 \qquad (\lambda>0) \tag{7}$$

and

$$BA^2B-2\lambda B^2+\lambda^2 \geqq 0 \qquad (\lambda>0). \tag{8}$$

Let $S=(BA^2B)^{\frac{1}{2}}$. Since

$$\mathfrak{D}(S^{-1}) = \mathfrak{D}((BA)^{-1}) \quad \text{and} \quad \|S^{-1}x\| = \|(BA)^{-1}x\|,$$

the spectral theory and (7) shows that

$$\|\{(S^2+\lambda^2)S^{-2}\}^{\frac{1}{2}}x\|^2 = \|x\|^2+\lambda^2\|S^{-1}x\|^2 = \|x\|^2+\lambda^2\|A^{-1}B^{-1}x\|^2 \geqq 2\lambda\|B^{-1}x\|^2,$$

so that

$$(S^2+\lambda^2)S^{-2} \geqq 2\lambda B^{-2}.$$

Then, as remarked in § 2,

$$2\lambda S^2(S^2+\lambda^2)^{-1} \leqq B^2.$$

Since, on the other hand, (8) implies

$$B^2 \leqq (2\lambda)^{-1}(S^2+\lambda^2),$$

Lemma shows $B^2=S$, hence $B^2=A^2$. This completes the proof.

J. Stampfli [4] proved that a hyponormal operator is normal if one of its powers is normal. We can generalize this result to paranormal case.

Theorem 6. *A paranormal operator T is normal if some power T^k is normal.*

Proof. First of all, recall that the compression of a paranormal operator to an invariant subspace is again paranormal, that every power of a paranormal operator is paranormal (cf. [1]) and that the spectral radius of a paranormal operator is equal to its norm (cf. [2]). Let $\{E(t)\}$ be the resolution of identity for the positive operator $(T^{*k}T^k)^{\frac{1}{k}}$. Since T commutes with the normal operator T^k, each $E(t)$ commutes with both T and T^* by the commutativity theorem. Now $TE(0)=0$, because $TE(0)$ is paranormal and

$$(TE(0))^k = T^k E(0) = 0.$$

It remains then to prove that $T(1-E(t))$ is normal for every $t>0$. Given $\varepsilon>0$, take a partition:

$$t = t_0 < t_1 < \cdots < t_n = \|T^k\| \quad \text{with} \quad 1-\varepsilon \leqq t_j^{-1}\cdot t_{j-1},$$

and let $E_j = E(t_j)-E(t_{j-1})$. Since

$$t_{j-1}\|E_j x\| \leqq \|T^k E_j x\| = \|T^{*k}E_j x\| \leqq t_j \|E_j x\|$$

and since TE_j is paranormal, it follows that

$$\|TE_j\| = \|T^* E_j\| \leqq t_j^{1/k}.$$

We shall show that

$$(1-\varepsilon)\|TE_j x\| \leqq \|T^* E_j x\| \leqq (1-\varepsilon)^{-1}\|TE_j x\|.$$

Suppose, for instance, that there is $y=E_j y$ such that $\|y\|=1$ and

$$\|Ty\| < (1-\varepsilon)\|T^* y\|.$$

Then

$$t_{j-1} \leqq \|T^k y\| \leqq \|TE_j\|^{k-1}\|Ty\| < (1-\varepsilon)t_j,$$

contradicting the choice of t_j. Now let $x = (1-E(t))x$, then

$$(1-\varepsilon)^2\|Tx\|^2 = (1-\varepsilon)^2 \sum_{j=1}^{n} \|TE_j x\|^2 \leqq$$

$$\leqq \|T^* x\|^2 \leqq (1-\varepsilon)^{-2} \sum_{j=1}^{n} \|TE_j x\|^2 = (1-\varepsilon)^{-2}\|Tx\|^2,$$

so that $\|Tx\| = \|T^* x\|$ follows as $\varepsilon \to 0$. This completes the proof.

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