On the essential numerical range, the essential spectrum, and a problem of Halmos

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Introduction. The first four sections of this paper are essentially a survey of what is known about the nature of the spectrum of a coset in the Calkin algebra. In these sections we show that a great deal of information can be obtained from a simple but not very well known theorem of F. WOLF [18]. That result (Theorem (1. 1)) gives several characterizations of those cosets that have a left inverse.

In section 2 we use Wolf's theorem to exhibit the relations between the spectrum and two different essential spectra of bounded operator.

Wolf's theorem immediately suggests introduction of the left essential spectrum of an operator A. This set turns out to coincide with the collection of Weyl limit points of the spectrum of A. It is thus of interest to know that it is non-empty as we show in section 3. In that section we also indicate the relations between the left essential spectrum, the boundary of the spectrum, and the approximate point spectrum of an operator.

In section 4 we use Wolf's theorem to obtain a description of the essential spectrum of a hyponormal coset in terms of eigenvalues.

In section 5 we obtain an analogue of Wolf's theorem for the numerical range of a coset. This result yields several new characterizations of the essential numerical range of an operator introduced in [17].

Finally, in section 6 we use the techniques of § 3 to show that the non-cyclic operators are norm-dense in $\mathfrak{B}(\mathfrak{H})$. This answers a question raised by HALMOS in [8].

Notation. In the following $\mathfrak{H}$ will denote a complex separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathfrak{H})$ denotes the algebra of all bounded linear operators on $\mathfrak{H}$, and $\mathfrak{K}$ denotes the ideal of compact operators on $\mathfrak{H}$. We shall let ν denote the canonical homomorphism from $\mathfrak{B}(\mathfrak{H})$ onto the *Calkin algebra* $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. (See [2].) The range,

*) The authors gratefully acknowledge the support of the National Science Foundation.

null space, and spectrum of an operator A are denoted by $\mathfrak{R}(A)$, $\mathfrak{N}(A)$, and $\sigma(A)$ respectively.

If $\mathfrak{G}$ is a complex Banach algebra with an identity of norm 1 then a *state* on $\mathfrak{G}$ is by definition a linear functional f on $\mathfrak{G}$ with the property $f(1)=1=\|f\|$. States always separate points of $\mathfrak{G}$ and if $\mathfrak{G}$ is a C^*-algebra then every state on $\mathfrak{G}$ is positive, i.e., $f(x^*x)\geqq 0$ for all $x\in\mathfrak{G}$.

1. Operators with closed range. Let T be a closed linear transformation with domain $\mathfrak{D}_T$ dense in $\mathfrak{H}$. F. WOLF [18] has shown that the following four conditions are equivalent:

(1) There exists a sequence $\{x_n\}$ of unit vectors in $\mathfrak{D}_T$ such that $x_n\to 0$ weakly and $Tx_n\to 0$ strongly.

(2) There exists an orthonormal sequence $\{e_n\}$ in $\mathfrak{D}_T$ such that $Te_n\to 0$ strongly.

(3) $E[0,\delta]\mathfrak{H}$ is infinite-dimensional for all $\delta>0$, where E is the spectral resolution of $(T^*T)^{\frac{1}{2}}$.

(4) Either the range $\mathfrak{R}(T)$ of T is non-closed, or the null-space $\mathfrak{N}(T)$ is infinite dimensional.

Consider the further conditions:

(5) Either $\mathfrak{N}(T)$ is infinite dimensional or 0 is a cluster point of $\sigma((T^*T)^{\frac{1}{2}})$.

(6) There exists an infinite-dimensional projection P such that $P\mathfrak{H}\subset\mathfrak{D}_T$ and TP is compact.

(7) There does not exists $X\in\mathfrak{B}(\mathfrak{H})$ such that $XT-I$ is compact.

(8) For every $\delta>0$ there exists a closed infinite-dimensional subspace $\mathfrak{M}_\delta\subset\mathfrak{D}_T$ such that $\|Tx\|\leqq\delta\|x\|$ for all $x\in\mathfrak{M}_\delta$.

Theorem (1. 1). *Conditions* (1)—(8) *are equivalent.*[1])

Proof. That (1) implies (3) results from the following computation of WOLF's:

$$\|x_n-E[0,\delta]x_n\|^2=\int_\delta^\infty d(E(t)x_n,x_n)=\int_\delta^\infty (t/\delta)^2 d(E(t)x_n,x_n)\leqq$$

$$\leqq(1/\delta)^2\int_\delta^\infty t^2 d(E(t)x_n,x_n)\leqq(1/\delta)^2\|(T^*T)^{\frac{1}{2}}x_n\|^2=(1/\delta)^2\|Tx_n\|^2\to 0.$$

The implications (3)→(2)→(1) are elementary. To see that (3) implies (6), choose an orthonormal sequence $\{e_n\}$ with $e_n\in E[0,1/n]\mathfrak{H}$, and let P be the projection on the span of the e_n. Then

$$P\mathfrak{H}\subset E[0,1]\mathfrak{H}\subset\mathfrak{D}_{(T^*T)^{\frac{1}{2}}}=\mathfrak{D}_T;$$

[1]) That (3) implies (6) was independently observed by C. APOSTOL (private communication).

if P_n is the projection on the span of $e_1, e_2, \ldots, e_n$, we also have

$$\|TP - TP_n\| \leqq 1/(n+1),$$

which implies that TP is compact.

Suppose that $K = XT - I$ is compact, $X \in \mathfrak{B}(\mathfrak{H})$. If P is a projection with $P\mathfrak{H} \subset \mathfrak{D}_T$ and TP compact, then because $XTP - P = KP$, P must be compact and therefore finite-dimensional. Thus (6) implies (7). To show that (7) implies (4), assume that (7) holds and that $\mathfrak{R}(T)$ is closed. Define the linear transformation $X: \mathfrak{R}(T) \to \mathfrak{N}(T)^\perp$ to be the inverse of T, and let $X = O$ on $\mathfrak{R}(T)^\perp$. Then X is closed and everywhere defined, hence bounded, and $I - XT$ is the projection on $\mathfrak{N}(T)$. It follows from (7) that $\mathfrak{N}(T)$ is infinite-dimensional.

Next, we show that (4) implies (3). This is clear if $\mathfrak{N}(T)$ is infinite-dimensional, so assume that $\mathfrak{R}(T)$ is not closed. If $T = U(T^*T)^{\frac{1}{2}}$ is the polar decomposition, it is well known that U carries $\mathfrak{R}((T^*T)^{\frac{1}{2}})$ isometrically onto $\mathfrak{R}(T)$, and so $\mathfrak{R}((T^*T))^{\frac{1}{2}}$ is non-closed. It follows that $E[0, \delta]\mathfrak{H}$ is is infinite-dimensional for every $\delta > 0$.

If (8) holds then for each integer $n \geqq 1$ there is a closed infinite-dimensional subspace $\mathfrak{M}_n \subset \mathfrak{D}_T$ such that $\|Tx\| \leqq n^{-1}\|x\|$ for all $x \in \mathfrak{M}_n$. By induction one can choose $e_n \in \in \mathfrak{M}_n$ such that $\|e_n\| = 1$, $0 = (e_n, P_{\mathfrak{m}_n} e_k) = (e_n, e_k)$ for $k < n$. Then $\{e_n\}$ is an orthonormal sequence and $\|Te_n\| \leqq 1/n$. Thus (8) implies (2).

The equivalence of (3) and (5) is a consequence of a well-known theorem of WEYL (Theorem (3. 4) below).

Finally, (3) implies (8) because $(T^*T)^{\frac{1}{2}}$, and therefore also T, is bounded by δ on $E[0, \delta]\mathfrak{H}$.

An operator $A \in \mathfrak{B}(\mathfrak{H})$ is called *semi-Fredholm* (or *Fredholm*) if the range $\mathfrak{R}(A)$ of A is closed and if at least one (both) of the subspaces $\mathfrak{N}(A)$, $\mathfrak{R}(A)^\perp$ is finite-dimensional. For a bounded operator A the equivalence of conditions (4) and (6) reduces to the assertion that A has closed range and finite-dimensional null space if and only if the coset $\nu(A)$ has a left inverse in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. The following are therefore immediate consequences:

Corollary 1 (Atkinson's Theorem). *$A \in \mathfrak{B}(\mathfrak{H})$ is a Fredholm operator if and only if $\nu(A)$ is invertible in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$.*

Corollary 2. *If A is semi-Fredholm (or Fredholm) then $A+K$ is semi-Fredholm (or Fredholm) for any compact operator K.*

Corollary 3. *The semi-Fredholm (or Fredholm) operators form an open set in $\mathfrak{B}(\mathfrak{H})$.*

Proof. Since A is Fredholm if and only if both A and A^* are semi-Fredholm, it suffices to prove that the set of those A in $\mathfrak{B}(\mathfrak{H})$ for which $\mathfrak{R}(A)$ is closed and $\mathfrak{N}(\mathfrak{H})$ is finite-dimensional is open. This in turn is a consequence of continuity of the quotient map ν and the fact that the left-invertible elements form an open set in any Banach algebra. (See [11] for example.)

We conclude this section with two related results. (The first is easy to prove; the second may be found in [6].)

Theorem (1. 2). *Suppose that A is a bounded operator with finite-dimensional null space. Then $\mathfrak{R}(A)$ is closed if and only if A maps closed bounded sets onto closed bounded sets.*

Theorem (1. 3). *Let A be a bounded linear transformation from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$. If the range of A is not closed in $\mathfrak{Y}$, then for each $\varepsilon > 0$ there is an infinite-dimensional closed subspace $\mathfrak{M}(\varepsilon)$ of $\mathfrak{X}$ such that the restriction of A to $\mathfrak{M}(\varepsilon)$ is compact and has norm less than ε.*

2. Essential spectra. There are several distinct definitions of the essential spectrum of an operator $A \in \mathfrak{B}(\mathfrak{H})$. In this section we shall indicate the basic facts concerning two of these. By definition, the *Wolf* (or Fredholm or Calkin) *essential spectrum* of A is the complement of the set of λ for which $A-\lambda$ is a Fredholm operator. Atkinson's theorem implies that the Wolf essential spectrum of A is therefore $\sigma(\nu(A))$, the spectrum of the coset $\nu(A)$ that contains A in the Calkin algebra. The second notion of essential spectrum that we shall examine is by definition the largest subset of $\sigma(A)$ that is invariant under compact perturbations of A, i.e., the set $\bigcap_{K \in \mathfrak{K}} \sigma(A+K)$ sometimes called the *Weyl spectrum*).[2]

In order to describe the relation between these two concepts we need to recall that a Fredholm operator has an *index* given by

$$\operatorname{ind}(A) = \dim \mathfrak{N}(A) - \dim \mathfrak{R}(A)^{\perp}$$

and that the index is invariant under compact perturbations (see [5, 6]). The following theorem is due to M. SCHECHTER [13]:

Theorem (2. 1).

$$\bigcup_{K \in \mathfrak{K}} \sigma(A+K) = \sigma(\nu(A)) \cup \{\lambda : A-\lambda \textit{ is Fredholm and } \operatorname{ind}(A-\lambda) \neq 0\}.$$

Proof. The quotient map ν is an algebra homomorphism, hence $\sigma(\nu(A)) = \sigma(\nu(A+K)) \subset \sigma(A+K)$ for every compact operator K. Moreover, if $A-\lambda$ is Fredholm with nonzero index, then so is $A+K-\lambda$ for any compact operator K. In particular, $A+K-\lambda$ is not invertible. This proves that the set on the right in Theo-

[2]) For a nice discussion of this topic, see S. K. BERBERIAN, The Weyl spectrum of an operator, *Indiana Univ. Math. J.*, **20** (1970), 529—554.

rem (2. 1) is contained in the set on the left. On the other hand, if λ does not belong to the set on the right, then $A-\lambda$ is Fredholm with index 0. This implies that $A-\lambda$ is of the form $B+K$ where B is invertible and K is compact. Thus $\lambda \notin \sigma(A-K)$ and hence λ does not belong to the set on the left in Theorem (2. 1).

The next two results indicate the relation between the essential spectrum and spectrum of an operator. Here $\sigma_p(T)$ denotes the point spectrum (=eigenvalues) of T.

Theorem (2. 2). $\sigma(A) = \bigcap \sigma(A+K) \cup \sigma_p(A)$.

Proof. The set on the right is clearly contained in $\sigma(A)$. Suppose then that $\lambda \in \sigma(A)$ and that $\lambda \notin \sigma(A+K)$ for some compact K. Then

$$(A+K-\lambda)(1-(A+K-\lambda)^{-1}K) = A-\lambda$$

is not invertible, so that $1-(A+K-\lambda)^{-1}K$ is not invertible. Hence 1 is an eigenvalue of the compact operator $(A+K-\lambda)^{-1}K$. But if $(A+K-\lambda)^{-1}Kx = x$ with $x \neq 0$, then $Kx = (A+K-\lambda)x$, and so $0 = (A-\lambda)x$. In other words, $\lambda \in \sigma_p(A)$. This completes the proof.

Theorem (2. 3). $\sigma(A) = \sigma(\nu(A)) \cup \sigma_p(A) \cup \sigma_p(A^*)^-$, *where the bar denotes complex conjugate.*

Proof. Suppose that $\lambda \in \sigma(A)$ and $\lambda \notin \sigma_p(A) \cup \sigma_p(A^*)^-$. Then $A-\lambda$ is one-to-one with dense range. Since $A-\lambda$ is not invertible, it follows that $\mathfrak{R}(A-\lambda)$ is not closed. Therefore $A-\lambda$ is not Fredholm so that $\lambda \in \sigma(\nu(A))$.

Remark. It is easy to see that if U is the unilateral shift of multiplicity 1, then $\bigcap \sigma(U+K)$ is the closed unit disk and $\sigma(\nu(U))$ is the unit circle. The larger essential spectrum is therefore obtained from the smaller one by filling in the hole. This is a general fact:

Theorem (2. 4). $\bigcap \sigma(A+K)$ *consists of* $\sigma(\nu(A))$ *together with some of the holes in* $\sigma(\nu(A))$.

Proof. Recall first that by definition a hole in a compact set X is a bounded component of the complement of X. We will use the following elementary fact: *If E and F are compact subsets of the plane such that $E \subset F$ and $\partial F \subset E$, then F is the union of E and those holes of E that meet F.*

Now if $E = \sigma(\nu(A))$ and $F = \bigcap \sigma(A+K)$, then $F-E$ consists of those complex numbers λ for which $A-\lambda$ is Fredholm of index $\neq 0$. By continuity of the index (see [5, 6]) this is an open set. Hence $\partial F \subset E$.

Corollary. $\bigcap \sigma(A+K)$ *and* $\sigma(\nu(A))$ *have the same convex hull.*

3. The left essential spectrum. Wolf's theorem motivates consideration of the left essential spectrum $\sigma_l(\nu(A))$ of an operator $A \in \mathfrak{B}(\mathfrak{H})$. By definition, a complex number λ belongs to $\sigma_l(\nu(A))$ if and only if the coset $\nu(A-\lambda) = \nu(A)-\lambda$ fails to have a left inverse in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. Equivalently, λ belongs to the left essential spectrum of A if and only if there is a sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly and $\|(A-\lambda)x_n\| \to 0$. Moreover, the x_n can even be chosen orthonormal (Theorem (1. 1).) In the special case in which A is self-adjoint the same concept was introduced by Weyl; accordingly such a complex number λ is also called a Weyl limit point of the spectrum of A (see [12]). The *right essential spectrum* $\sigma_r(\nu(A))$ is defined in the obvious way.

The concepts just introduced derive their usefulness from the following:

Theorem (3. 1). $\sigma_l(\nu(A)) \cap \sigma_r(\nu(A)) \supset \partial\sigma(\nu(A))$. *Hence* $\sigma_l(\nu(A))$ *is a non-empty compact subset of* $\sigma(\nu(A))$.

Proof. The theorem is an immediate consequence of two well-known facts about Banach algebras. First, the set $\mathfrak{G}_l$ of elements that have a left inverse is open and second, any point of the boundary of $\mathfrak{G}_l$ is a right topological divisor of 0. (See [11] for example.)

Theorem (3. 2). *If* $A \in \mathfrak{B}(\mathfrak{H})$, *then* $\pi(A)$, *the approximate point spectrum of* A, *consists of* $\sigma_l(\nu(A))$ *together with the eigenvalues of finite multiplicity.*

Proof. If $\lambda \in \pi(A)$ and $\lambda \notin \sigma_l(\nu(A))$, then $A-\lambda$ is not bounded below but has closed range and finite dimensional null space. Hence $\mathfrak{N}(A-\lambda) \neq 0$ so that λ is an eigenvalue of finite multiplicity.

The next result of PUTNAM [9] is much deeper: [3])

Theorem (3. 3). *If* $A \in \mathfrak{B}(\mathfrak{H})$ *and* $\lambda \in \partial\sigma(A)$, *then either* λ *is an isolated point of* $\sigma(A)$ *and an eigenvalue of finite multiplicity, or it belongs to* $\sigma_l(\nu(A))$, *that is there is an orthonormal sequence* $\{e_n\}$ *such that* $\|(A-\lambda)e_n\| \to 0$.

If A is an operator with no eigenvalues, then Theorem (2. 2) asserts that $\sigma(A) = \cap\,\sigma(A+K)$. In particular, each compact perturbation of A has a larger spectrum that that of A. There is a simple relationship between these spectra:

Corollary. *Let* $A \in \mathfrak{B}(\mathfrak{H})$ *and assume that* A *has no eigenvalues. Then for any compact operator* K

$$\sigma(A+K) = \sigma(A) \cup \mathfrak{J} \cup \text{ some holes in } \sigma(A),$$

where $\mathfrak{J}$ *is the set of isolated eigenvalues of* $A+K$ *of finite multiplicity.*

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[3]) Prof. PUTNAM has requested that we refer to this result as the Putnam—Schechter theorem.

Proof. We have $\sigma(A)\cup\mathfrak{J}\subset\sigma(A+K)$ (by the preceding corollary). Also, by Putnam's Theorem,

$$\partial\sigma(A+K)\subset\sigma_l(A+K)\cup\mathfrak{J}=\sigma_l(\nu(A))\cup\mathfrak{J}\subset\sigma(A)\cup\mathfrak{J}.$$

The proof is completed by an application of the topological fact used in the proof of Theorem (2. 4).

If A is a normal operator on $\mathfrak{H}$, then it is easy to see from Theorem (2. 1) that $\bigcap\sigma(A+K)=\sigma(\nu(A))$. Moreover, $\sigma_l(\nu(A))=\sigma(\nu(A))$, and if E is the spectral measure of A, then $\lambda\in\sigma(\nu(A))$ if and only if every neighborhood $\mathfrak{U}$ of λ has infinite spectral measure ($\dim\mathfrak{R}(E(\mathfrak{U}))=\infty$). From this it is easy to obtain WEYL's characterization of the essential spectrum of A (see [12, p. 367]).

Theorem (3. 4). *If A is normal, then $\sigma(\nu(A))=\sigma_l(\nu(A))$ consists of the cluster points of $\sigma(A)$ together with the isolated eigenvalues of A of infinite multiplicity.*

Weyl's theorem has recently been generalized to hyponormal operators by COBURN [3]:

Theorem (3. 5). *If A is hyponormal, then $\bigcap\sigma(A+K)$ consists of the cluster points of $\sigma(A)$ and the isolated eigenvalues of infinite multiplicity.*

4. Eigenvalues in the Calkin algebra. In this section we obtain more detailed information about the Wolf essential spectrum of special operators. For statements about elements in the Calkin algebra it is convenient to use lower case Latin letters a, p instead of the more cumbersome notation $\nu(A)$, $\nu(P)$ for the cosets containing the operator A, P.

We begin with a simple reformulation of part of Wolf's theorem:

Theorem (4. 1). *Let $a\in\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$ and let $\lambda\in\sigma(a)$. Then there is a projection $p\neq0$ such that either $ap=\lambda p$ or $pa=\lambda p$.*

Proof. Suppose λ belongs to $\sigma_l(a)$. If $A\in a$, then $\nu(A-\lambda)$ does not have a left inverse, hence (Theorem 1. 1) there is a compact operator K such that $\dim\mathfrak{N}(A-\lambda-K)=\infty$. Let P be the orthogonal projection onto $\mathfrak{N}(A-\lambda-K)$ and let $p=\nu(P)$. Then $(A-\lambda-K)P=O$ so that $(a-\lambda)p=0$. Moreover, $p\pm0$ since P has infinite rank.

To complete the proof we must also consider the possibility that $\nu(A-\lambda)$ fails to have a right inverse. However on taking adjoints, this case reduces to the one just discussed.

Corollary. *If $A\in\mathfrak{B}(\mathfrak{H})$ then there are orthogonal projections P and Q of infinite rank and nullity and a complex number λ such that $(A-\lambda)P$ and $Q(A-\lambda)$ are compact.*

Proof. Any two projections of infinite rank contain orthogonal sub-projections of infinite rank and nullity. Projections with the asserted properties can therefore be found for any $\lambda \in \sigma_l(\nu(A)) \cap \sigma_r(\nu(A))$.

Note that if P is as in the corollary, then $\nu(A)\nu(P)=\nu(P)\nu(A)\nu(P)$. Thus $AP-PAP$ is compact so that A has an invariant subspace "modulo the compacts".

Theorem (4. 1) shows that λ belongs to the Wolf essential spectrum of A if and only if either λ is an eigenvalue of (the regular representation of) $\nu(A)$ or $\bar{\lambda}$ is an eigenvalue of $\nu(A^*)$. We will shapren this assertion for hyponormal elements of the Calkin algebra. For this we need a lemma.

Lemma 4. 1. *Let $\mathfrak{G}$ be a C^*-algebra with unit and let A be a hyponormal element of $\mathfrak{G}$ (i.e., $A^*A \geqq AA^*$). If A has a right inverse, then A is invertible.*

Proof. Since A is hyponormal, $Z^*AA^*Z \leqq Z^*A^*AZ$ for any Z. Hence $AZ=O$ implies $A^*Z=O$. Suppose now that $AX=1$. Then $A(XA-1)=O$ so $A^*(XA-1)=O$ and thus $XA-1 = X^*A^*(XA-1) = O$.

Theorem (4. 2). *Let a be a hyponormal element of $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. Then*

(1) *$\lambda \in \sigma(a)$ if and only if there is a projection $p \neq 0$ such that $a^*p=\bar{\lambda}p$.*

(2) *If p is a projection such that $ap=\lambda p$, then $a^*p=\bar{\lambda}p$.*

(3) *If $ap_i=\lambda_i p_i$ for $i=1, 2$, and if $\lambda_1 \neq \lambda_2$, then $p_1p_2=0$.*

Proof. The first assertion is an obvious consequence of Lemma (4. 1). To prove (2), note that the condition $(a-\lambda)p = 0$ implies $(a-\lambda)^*p = 0$ because $a-\lambda$ is hyponormal (see the proof of Lemma (4. 1)).

If $ap_1=\lambda_1p_1$ and $ap_2=\lambda_2p_2$ then $p_1a=\lambda_1p_1$ by (2) so that $\lambda_2p_1p_2=p_1ap_2=$ $=\lambda_1p_1p_2$.

Remarks. (1) It is easy to exhibit non-normal hyponormal elements of $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$. For example, if U is an isometry with infinite defect, then $\nu(U)$ is such an element. Or again, if B is a positive noncompact operator with $0 \in \sigma(\nu(B))$, then (Radjavi [10]) there exists $A \in \mathfrak{B}(\mathfrak{H})$ such that $A^*A-AA^* = B$. The coset $\nu(A)$ is then hyponormal but not normal.

(2) It is not true that every hyponormal coset a contains an operator of the form hyponormal+compact. (Let $a=\nu(A)$ where A is the adjoint of the unilateral shift and compute the Fredholm index.)

It is well known that any eigenvector of an operator A corresponding to an eigenvalue λ of modulus $|\lambda|=\|A\|$ must reduce A, i.e., $Ax=\lambda x$ implies $A^*x=\bar{\lambda}x$. The next result is the analogue of this fact for the Calkin algebra:

Theorem (4. 3). *Let $a \in \mathfrak{B}(\mathfrak{H})/\mathfrak{K}$ and suppose that there is a $\lambda \in \sigma_l(a)$ with $|\lambda|=$ $=\|a\|$. If p is a projection such that $ap=\lambda p$, then $a^*p=\bar{\lambda}p$.*

Proof. Without loss of generality we may suppose that $\|a\|=1=\lambda$. Then $pa^*p=(pap)^*=p$, and so

$$0 \leqq (a^*p = p)^*(a^*p-p) = paa^*p-pap-pa^*p+p = paa^*p-p = p(aa^*-1)p \leqq 0.$$

Hence $a^*p=p$.

Corollary. *If $A\in\mathfrak{B}(\mathfrak{H})$ and if the coset $\nu(A)$ has norm equal to its spectral radius, then there exists a projection P of infinite rank and a complex number λ such that $(A-\lambda)P$ and $P(A-\lambda)$ are both compact.*

Remark. A coset $\nu(A)$ has norm equal to its spectral radius in each of the following cases:

(i) $\nu(A)$ is hyponormal.

(ii) $\nu(A)$ contains a Toeplitz operator.

(iii) A has norm equal to its spectral radius and A has no isolated eigenvalues of finite multiplicity.

(Sufficiency of (i) can be proved by a slight modification of the proof in [14] of the corresponding fact in $\mathfrak{B}(\mathfrak{H})$. Condition (iii) is sufficient by Putnam's Theorem (3. 3), and (ii) is a special case of (iii) since a Toeplitz operator has no isolated eigenvalues of finite multiplicity [3].)

The corollary therefore implies that if A is a compact perturbation of a hyponormal operator or a Toeplitz operator then there is an orthonormal sequence $\{e_n\}$ and a complex number λ such that

$$(A-\lambda)e_n \to 0, \quad (A^*-\bar{\lambda})e_n \to 0.$$

Consequently, A is uniformly approximable by operators with a reducing eigenvector. (See [15], Theorems 1, 2.)

5. The essential numerical range. The numerical range of a bounded operator A on $\mathfrak{H}$ is defined as

$$W(A) = \{(Ax, x): \|x\| = 1\}.$$

In [17] a generalized *numerical range* $W_0(a)$ was introduced for an element a of an arbitrary complex Banach algebra $\mathfrak{G}$ with norm 1 unit. By definition $W_0(a)$ consists of the complex numbers of the form $f(a)$ where f ranges over the states of $\mathfrak{G}$. The set $W_0(a)$ is convex, compact, and contains the spectrum of a. If $\mathfrak{G}$ is a subalgebra of $\mathfrak{B}(\mathfrak{H})$ then for $A\in\mathfrak{G}$ the numerical range $W_0(A)$ coincides with the closure $W(A)^-$ of the usual numerical range.

The *essential numerical* range [17] of an $A\in\mathfrak{B}(\mathfrak{H})$ is by definition the numerical range $W_0(\nu(A))$ of the coset in $\mathfrak{B}(\mathfrak{H})/\mathfrak{K}$ that contains A. We shall denote this set by $W_e(A)$ in the following. A more explicit identification is given by the formula

$$W_e(A) = \bigcap \{W(A+K)^-: K\in\mathfrak{K}\}.$$

(Thus, unlike the situation for the essential spectrum, there is only one "natural" definition of essential numerical range. See Theorem (2. 1).)

The preceding formula was proved in [17] by a convexity argument. A simpler proof is obtained by noting that a complex number λ belongs to either side if and only if $\|A+K-z\| \geqq |\lambda-z|$ for all complex numbers z and all compact operators K. (This is an immediate consequence of Theorem 4 of [17] and the definition of the norm of the coset $\nu(A-z)$.)

In this section we give several new descriptions of the essential numerical range. These are obtained from the following analogue of Theorem (1. 1):

Theorem (5. 1). *For $T\in\mathfrak{B}(\mathfrak{H})$, the following conditions are equivalent:*

(1) $0\in\bigcap\{W(T+F)^- : F$ is of finite rank$\}$.

(2) $0\in W_e(T)$.

(3) *There exists a sequence $\{x_n\}$ of unit vectors such that $x_n\to 0$ weakly and $(Tx_n, x_n)\to 0$.*

(4) *There exists an orthonormal sequence $\{e_n\}$ such that $(Te_n, e_n)\to 0$.*

(5) *There exists an infinite-dimensional projection P such that PTP is compact.*

Proof. The implications $(5)\to(4)\to(3)\to(2)\to(1)$ are clear. We first prove that (1) implies (4). Let $\varepsilon_k\to 0$, and assume that orthogonal unit vectors $e_1, e_2, \dots, e_n$ have been found so that $|(Te_k, e_k)|<\varepsilon_k$ for $k=k, 2, \dots, n$. Let $\mathfrak{M}$ be the subspace spanned by the e_k, and let P be the projection on $\mathfrak{M}$. In order to exhibit a unit vector e_{n+1} orthogonal to $\mathfrak{M}$ with $|(Te_{n+1}, e_{n+1})|<\varepsilon_{n+1}$, it is sufficient to show that $0\in W((I-P)T|\mathfrak{M}^\perp)^-$. To see that the latter condition holds, choose

$$\mu\in W((I-P)T|\mathfrak{M}^\perp),$$

and let

$$F = \mu P - PTP-(I-P)TP-PT(I-P).$$

Then F is of finite rank, and

$$T+F = \mu P+(I-P)\,T(I-P) = \mu I_{\mathfrak{M}}\oplus(I-P)T|\mathfrak{M}^\perp.$$

In general it is true that $W(A\oplus B)$ is the convex hull of $W(A)$ and $W(B)$; and thus it follows that

$$W(T+F) = W((I-P)T|\mathfrak{M}^\perp)$$

since $W((I-P)T|\mathfrak{M}^\perp)$ is convex and contains μ. Hence (1) implies $0\in W((I-P)T|\mathfrak{M}^\perp)$, as required.

To complete the proof we show that (4) implies (5). Let $\{e_n\}$ be an orthonormal sequence with $(Te_n, e_n)\to 0$. By passing to a subsequence we can assume that

$$(*)\qquad \sum_{n=1}^{\infty} |(Te_n, e_n)|^2 < \infty.$$

Put $n_1=1$. Since

$$\sum_{n=1}^{\infty} |(Te_{n_1}, e_n)|^2 \leqq \|Te_{n_1}\|^2 \quad \text{and} \quad \sum_{n=1}^{\infty} |(Te_n, e_{n_1})|^2 \leqq \|T^* e_{n_1}\|^2$$

by Bessel's inequality, there is an integer $n_2>n_1$ such that

$$\sum_{n=n_2}^{\infty} |(Te_{n_1}, e_n)|^2 < \tfrac{1}{2} \quad \text{and} \quad \sum_{n=n_2}^{\infty} |(Te_n, e_{n_1})|^2 < \tfrac{1}{2}.$$

By iterating this procedure we generate a strictly increasing sequence $\{n_k\}$ of positive integers with the property

$$(**) \qquad \sum_{n=n_{k+1}}^{\infty} |(Te_{n_k}, e_n)|^2 < 2^{-k} \quad \text{and} \quad \sum_{n=n_{k+1}}^{\infty} |(Te_n, e_{n_k})|^2 < 2^{-k}$$

for all $k\geqq 1$. Inequalities $(*)$ and $(**)$ imply that

$$\sum_{i,j=1}^{\infty} |(Te_{n_i}, e_{n_j})|^2 < \infty.$$

If P is the projection on the span of the e_{n_k}, this means that PTP is a Hilbert—Schmidt operator, and therefore PTP is compact.

Corollary. *Each of the following conditions is necessary and sufficient in order that* $\lambda \in W_e(T)$:

(1) $(Tx_n, x_n) \to \lambda$ *for some sequence* $\{x_n\}$ *of unit vectors such that* $x_n \to 0$ *weakly.*

(2) $(Te_n, e_n) \to \lambda$ *for some orthonormal sequence* $\{e_n\}$.

(3) $PTP-\lambda P$ *is compact for some infinite-dimensional projection* P.

Remarks. (1) As we observed in § 4 a point λ belongs to the Wolf essential spectrum of A if and only if $v(A)v(P)=\lambda v(P)$ or $v(P)v(A)=\lambda v(P)$ for some non-zero projection $v(P)$. By Theorem (5. 1) the corresponding statement for the numerical range is: $\lambda \in W(A)$ if and only if $v(P)v(A)v(P)=\lambda v(P)$ for some non-zero projection $v(P)$; i. e., $pap=\lambda p$.

(2) The analogy between Theorems (1. 1) and (5. 1) is not complete. Call a sequence of vectors *non-compact* if it has no strongly convergent subsequence. Then Wolf's statement of condition (3) of Theorem (1. 1) requires only that $Ax_n \to 0$ strongly for some non-compact sequence $\{x_n\}$ of unit vectors. However, the corresponding reformulation of condition (3) of Theorem (5. 1) is not equivalent to the conditions of that theorem. For example, let A be the operator with matrix diag $(-1, 1, 1, \ldots)$ in an orthonormal basis $\{e_n\}$ and let $x_n=(e_1+e_{n+1})/\sqrt{2}$. Then $0 \notin W_e(A)$, and yet $\{x_n\}$ is a non-compact sequence with $(Ax_n, x_n) \to 0$.

(3) If the space $\mathfrak{H}$ is finite-dimensional no sequence of unit vectors can converge weakly so that the conditions of the corollary of Theorem (5. 1) have no reasonable analogue. A possible replacement is the condition: $(Ae_i e_i)=\lambda$ for some orthonormal set $\{e_i: 1 \leqq i \leqq k\}$ where $k \leqq \dim \mathfrak{H}$. In [4] it is shown that the set of complex numbers λ with this property constitute the k-numerical range [7] of A.

(4) The preceding remark suggests the following question: If $A \in \mathfrak{B}(\mathfrak{H})$ for which complex numbers λ does there exist a projection P of infinite rank such that $P(A-\lambda)P = O$? A partial answer is given in [1].

(5) The Corollary of Theorem (5. 1) shows that if f is a state on $\mathfrak{B}(\mathfrak{H})$ that annihilates $\mathfrak{K}$, then for each $A \in \mathfrak{B}(\mathfrak{H})$ there is an orthonormal sequence $\{e_n\}$ such that $f(A) = \lim_n (Ae_n, e_n)$.

6. Non-cyclic operators. We conclude with an application to a problem recently proposed by HALMOS [8]: does the set of cyclic operators have a non-empty interior? Although this set is readily seen to be open when $\dim \mathfrak{H} < \infty$, we have:

Theorem (6. 1). *When $\mathfrak{H}$ is infinite-dimensional, the non-cyclic operators are norm-dense in $\mathfrak{B}(\mathfrak{H})$.*

Proof. We use the observation (HALMOS, op. cit.) that if $\mathfrak{N}(A^*)=\mathfrak{R}(A)^\perp$ has dimension at least two, then for each f the span of $f, Af, A^2f, \dots$ has codimension at least one, and so A is non-cyclic. Now let $T \in \mathfrak{B}(\mathfrak{H})$, choose λ in the left essential spectrum of T^* (cf. Theorem (3. 1) above), and let $\varepsilon > 0$. Then there exist orthogonal unit vectors φ and ψ such that $\|(T^*-\lambda I)\varphi\| < \varepsilon$ and $\|(T^*-\lambda I)\psi\| < \varepsilon$. Let P be the projection on the span of φ and ψ, and set

$$S^* = \lambda P + T^*(I-P).$$

Then $T^*-S^* = (T^*-\lambda I)P$ has norm at most 2ε. But $S^*-\lambda I$ has nullity at least two, so $S-\bar{\lambda}I$ is non-cyclic, and therefore S is non-cyclic. Since $\|T-S\| \leqq 2\varepsilon$ and ε is arbitrary, the proof is complete.

Remark. Since the approximating operator differs from the given operator only on a finite-dimensional subspace, the proof shows that the non-cyclic operators are dense in any norm.

For another application of Theorem (3. 1) we refer the reader to [16] where it is shown that for any $A \in \mathfrak{B}(\mathfrak{H})$ the range of the inner derivation $X \to AX-XA$ is never dense in $\mathfrak{B}(\mathfrak{H})$, nor does it contain all finite dimensional operators.

Our final result is an application of Wolf's theorem.

Theorem (6. 2) *Let $T \in \mathfrak{B}(\mathfrak{H})$. Then $1-T^*T$ is compact if and only if $T=U+K$ where K is compact and U is either an isometry or a co-isometry with finite-dimensional null space.*

Proof. Sufficiency is trivial. To prove necessity suppose that $1-T^*T$ is compact. Since $1-T^*T=(1+\sqrt{T^*T})(1-\sqrt{T^*T})$ and the left factor is invertible, it follows that $1-\sqrt{T^*T}$ is compact.

If $x_n\to 0$ weakly and $Tx_n\to 0$ strongly then $x_n=(1-T^*T)x_n+T^*Tx_n\to 0$ strongly Hence by Theorem (1. 1) T has closed range and finite dimensional null space.

Suppose $\dim \mathfrak{N}(T)\leqq\dim \mathfrak{R}(T)^\perp$. Replacing T by $T+K$ for some compact K if necessary we may assume that T is one-to-one. If $T=U\sqrt{T^*T}$ is the polar decomposition of T, then U is an isometry and $T=U-U(1-\sqrt{T^*T})=U+\text{compact}$.

To complete the proof we must also consider the case $\dim \mathfrak{R}(T)^\perp<\dim \mathfrak{N}(T)$. The above argument shows tha $T^*=U+K$ where U is an isometry and K is compact. The null space of U^* is finite-dimensional by the hypothesis on T.

Corollary. $T=\text{isometry}+\text{compact}$ *if and only if* $1-T^*T$ *is compact and* T *is semi-Fredholm with* $\operatorname{ind}(T)\leqq 0$.

Remarks. If $1-T^*T$ is compact and $1-TT^*$ is not compact then $\sigma(T)$ contains the unit disk. This is a consequence of the theorem and fact that in any C^*-algebra with identity the spectrum of a non-unitary isometry is the unit disk.

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(Received January 29, 1971)