Operators with essentially disconnected spectrum

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1. Introduction. Throughout this paper $\mathfrak{H}$ will denote an infinite dimensional complex Hilbert space, $\mathcal{L}(\mathfrak{H})$ will represent the algebra of all (bounded linear) operators on $\mathfrak{H}$, and by $\mathcal{K}$ we shall mean the ideal of all compact operators on $\mathfrak{H}$. Let π be the canonical projection from $\mathcal{L}(\mathfrak{H})$ onto the (Calkin) quotient algebra $\mathcal{L}(\mathfrak{H})/\mathcal{K}$. For every $T \in \mathcal{L}(\mathfrak{H})$ the spectrum $E(T)$ of $\pi(T)$ in $\mathcal{L}(\mathfrak{H})/\mathcal{K}$ will be called the Calkin essential spectrum of T.

Definition. We say that the spectrum $\Sigma(T)$ of an operator $T \in \mathcal{L}(\mathfrak{H})$ is essentially disconnected if the polynomial hull $\hat{\Sigma}(T)$ of $\Sigma(T)$ is disconnected and $E(T)$ intersects more than one component of $\hat{\Sigma}(T)$ (the polynomial hull $\hat{X}$ of a compact subset X of the complex plane $\mathbf{C}$ is the complement of the unbounded component of $\mathbf{C}-X$).

Our main purpose in this note is to initiate the study of the class of all operators whose spectrum is essentially disconnected, which we shall denote by (ED). Examples of operators having such a property are easy to come by, taking, for instance, the direct sum of two operators on $\mathfrak{H}$ whose spectra are far from each other. In particular, a self-adjoint operator has an essentially disconnected spectrum if and only if its essential spectrum is disconnected. Of course this is not the case for an arbitrary operator on $\mathfrak{H}$.

Operators in (ED) have many interesting properties, especially those concerned with perturbations by either small norm operators or compact ones. Thus, if $T \in (ED)$, then $T+K \in (ED)$ and $\hat{\Sigma}(T+K)$ is disconnected for every $K \in \mathcal{K}$. Furthermore, an operator $T \in (ED)$ if and only if $\hat{E}(T)$ is disconnected (Theorem 2). On the other hand, the class (ED) is open in the uniform topology of $\mathcal{L}(\mathfrak{H})$ (Theorem 7). We also prove (Theorem 8) that if $T \in (ED)$ and $\mathcal{I}_T$ denotes the lattice of invariant subspaces of T equipped with the topology induced by the distance between subspaces ([5]), then there exist two infinite dimensional subspaces $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{I}_T$ which

*) This paper consists of part of the author's doctoral dissertation written at the University of Michigan. The author would like to thank Prof. C. PEARCY for his encouragement during the preparation of this paper.

are isolated points of $\mathscr{I}_T$, such that $\mathscr{I}_T$ is homeomorphic to $\mathscr{I}_{T_1}\times\mathscr{I}_{T_2}$, where $T_j=T|\mathfrak{M}_j$, $j=1, 2$.

In proving the last result we need to show (see the proof of Theorem 8) that every $T\in(ED)$ is similar to the direct sum of two operators S_1, S_2 acting on infinite dimensional Hilbert spaces such that $\hat{\Sigma}(S_1)\cap\hat{\Sigma}(S_2)=\emptyset$. Therefore, up to similarity, every operator in (ED) looks like the example given above.

We begin our considerations (§ 2) by discussing some relations between the different kinds of essential spectra of an operator. One of our main results in this direction is proved in § 3 (Theorem 4) and states that each separated part of the different kind of essential spectra of an operator T is an upper-semicontinuous function of T.

Finally, in § 5 we enumerate some questions raised in the paper and we present partial answers to some of them. As an immediate byproduct of these results we derive interesting properties of hyponormal operators.

2. Some properties of the essential spectrum. To begin with we recall some facts from the theory of Fredholm operators ([8]). For $T\in\mathscr{L}(\mathfrak{H})$ we have that $\pi(T)$ is invertible in $\mathscr{L}(\mathfrak{H})/\mathscr{K}$ if and only if ran T is closed, $\alpha(T)=\dim \operatorname{null} T$ is finite and $\beta(T)=\dim \operatorname{null} T^*(=\dim(\operatorname{ran} T)^\perp=\alpha(T^*))$ is also finite (Atkinson's theorem). In this case T is called a Fredholm operator and its index is defined by $j(T)=$ $=\alpha(T)-\beta(T)$. Thus, the set Φ of all Fredholm operators is an open subset of $\mathscr{L}(\mathfrak{H})$ in the uniform topology; its components are also open and they correspond to each value of the (integer valued) function $j(T)$. We shall denote by Φ_0 the component of Φ consisting of all Fredholm operators of index zero.

With the above notation the Calkin essential spectrum of an operator T can be expressed as $E(T)=\{\lambda\in\Sigma(T): T-\lambda\notin\Phi\}$. Another important concept native to the theory of compact perturbation is the Weyl spectrum $\Omega(T)$ of T ([1], [3]) i.e. $\Omega(T)=\{\lambda\in\Sigma(T): T-\lambda\notin\Phi_0\}$. SCHECHTER proved ([13]) that $\Omega(T)=\bigcap_{K\in\mathscr{K}}\Sigma(T+K)$.

On the other hand, BROWDER introduced in [2] a third concept of essential spectrum, namely $B(T)=\Sigma(T)-\{\lambda\in\Sigma(T): T-\lambda\in\Phi_0,\ \lambda$ is an isolated point of $\Sigma(T)\}$.

Clearly $E(T)\subset\Omega(T)\subset B(T)$.

It is easy to see that if λ is an isolated point of $\Sigma(T)$ and $T-\lambda\in\Phi$, then $T-\lambda\in\Phi_0$. Also, it is an immediate consequence of [8], Chapter 4, Theorem 5. 31 that if λ is a limit point of $b\Sigma(T)$ (here and in what follows bX denotes the boundary of the set X), then $\lambda\in E(T)$. Therefore we conclude that $bB(T)\subset E(T)$. Given a compact subset X of the plane, a hole of X is a component of $\hat{X}-X$. If Y is another compact set such that $b(X)\subset Y\subset X$, it follows that $b(X)\subset b(Y)$, $\hat{X}=\hat{Y}$ and X can be obtained from Y by filling in some holes of Y. We sumarize all the above discussion in the following theorem:

Theorem 1. *Let* $T\in\mathscr{L}(\mathfrak{H})$. *Then*

a) $E(T)\subset\Omega(T)\subset B(T)$,

b) $bB(T)\subset b\Omega(T)\subset bE(T)$,

c) $\hat{E}(T)=\hat{\Omega}(T)=\hat{B}(T)$,

d) $\Omega(T)(B(T))$ *can be obtained from* $E(T)(\Omega(T))$ *by filling in some holes of* $E(T)(\Omega(T))$.*)

Corollary 2. 1. *Let* $T\in\mathscr{L}(\mathfrak{H})$. *If* $E(T)$ *is connected,* $\Omega(T)$ *is connected, and if* $\Omega(T)$ *is connected,* $B(T)$ *is connected.*

One can construct very easily examples showing that none of the reverse implications in Corollary 2. 1 hold in general. Let $\mathfrak{G}$ be a separable Hilbert space, and let V be a unilateral shift of multiplicity one on $\mathfrak{G}$; also let $N\in\mathscr{L}(\mathfrak{G})$ be any quasinilpotent operator. If we denote by D the closed unit disc in $\mathbf{C}$ we have $B(V\oplus V^*\oplus N)=D$, while $\Omega(V\oplus V^*\oplus N)=E(V\oplus V^*\oplus N)=bD\cup\{0\}$ ([7], Problem 144). Furthermore, $\Omega(V\oplus N)=D$, but $E(V\oplus N)=bD\cup\{0\}$.

Theorem 2. *For* $T\in\mathscr{L}(\mathfrak{H})$, *the following statements are equivalent:*

a) $T\in(ED)$, *that is* $\hat{\Sigma}(T)$ *is disconnected and* $E(T)$ *intersects more than one component of* $\hat{\Sigma}(T)$,

b) $\hat{E}(T)$ *is disconnected.*

Proof. The proof is a consequence of the fact that $\hat{\Sigma}(T)-\hat{E}(T)(=\hat{\Sigma}(T)-\hat{B}(T))$ consists of isolated points λ such that $T-\lambda\in\Phi_0$.

Next we introduce the following terminology: given a compact subset X of the plane we will denote by rad X the radius of X, i.e. $\operatorname{rad} X=\sup_{\lambda\in X}|\lambda|$. Theorem 1 tells us that $\operatorname{rad} E(T)=\operatorname{rad}\Omega(T)=\operatorname{rad} B(T)$. Thus it is natural to call this common value the essential spectral radius of T, which shall be denoted by $r_e(T)$.

Nussbaum in [9] already observed that the radius of the different kinds of essential spectra are the same, but our argument is much simpler than that used by Nussbaum.

The next lemma makes the definition of the essential spectral radius even more natural.

Lemma 2. 2. *If* $T\in\mathscr{L}(\mathfrak{H})$, *then*

$$r_e(T)=\inf_{K\in\mathscr{K}} r(T+K),$$

*) This interesting relationship between the Calkin spectrum and the Weyl spectrum is also discussed by Fillmore, Stampfli and Williams in their recent paper "Essential numerical range, essential spectrum and a problem of Halmos", *Acta Sci. Math.*, **33** (1972), 179—192.

where $r(T+K)$ denotes the spectral radius of $T+K$. Moreover, if Q is any projection in $\mathcal{L}(\mathfrak{H})$ and T_Q denotes the compression of T to the range of Q, i.e. $T_Q=(QT)|\text{ran } Q$ then we also have

$$r_e(T) = \inf_{(1-Q)\in\mathcal{P}_f} r(T_Q),$$

where $\mathcal{P}_f$ is the set of all finite rank projections in $\mathcal{L}(\mathfrak{H})$.

Proof. Let $\lambda_0\in\Sigma(T)$ be an isolated point such that $T-\lambda_0\in\Phi_0$. Set $\Sigma_0 = \Sigma(T)-\{\lambda_0\}$ and E_{Σ_0} the idempotent associated with the clopen subset Σ_0 of $\Sigma(T)$ ([11], § 148). Also we denote by Q_0 the (orthogonal) projection onto ran E_{Σ_0}. It follows that $\Sigma_0=\Sigma(T_{Q_0})$; hence $\Sigma(TQ_0) = \Sigma(T_{Q_0})\cup\{0\} = \Sigma_0\cup\{0\}$. Therefore $\text{rad}\,(\Sigma(TQ_0))=\text{rad}\,(\Sigma_T Q_0)$. Since $1-Q_0$ is a finite rank projection we see that

$$\inf_{K\in\mathcal{K}} r(T+K) \leqq r(TQ_0) = \text{rad}\,\Sigma_0 \quad \text{and also} \quad \inf_{(1-Q)\in\mathcal{P}_f} r(T_Q) \leqq \text{rad}\,\Sigma_0.$$

Now the same argument used for λ_0 can be applied to any set consisting of finitely many isolated $\lambda\in\Sigma(T)$ such that $T-\lambda\in\Phi_0$. In this way we conclude that

$$\inf_{K\in\mathcal{K}} r(T+K) \leqq \text{rad}\,B(T) = r_e(T), \quad \text{and} \quad \inf_{(1-Q)\in\mathcal{P}_f} r(T_Q) \leqq r_e(T),$$

proving half of the lemma. On the other hand, recalling that $r_e(T)=\text{rad}\,\Omega(T)$, $\Omega(T) = \bigcap_{K\in\mathcal{K}} \Sigma(T+K)$ and observing that $\Omega(T) \subset \bigcap_{(1-Q)\in\mathcal{P}_f} \Sigma(T_Q)$ we see that the other half is also valid.

Remark. We list below some other elementary properties of the essential spectral radius of an operator T.

i) It follows from $\text{rad}\,E(T)=r_e(T)$ that

$$r_e(T) = \lim_{n\to\infty} \|\pi(T^n)\|^{1/n}.$$

ii) From Lemma 2. 2, and [7], Problem 122 it is not hard to see that

$$r_e(T) = \inf_{S\in\Phi_0'} \|[\pi(S)]^{-1}\pi(T)\pi(S)\|.$$

iii) Let $w_e(T)$ be the essential numerical radius of T ([12], § 3) that is $w_e(T)= \text{rad}\,W_e(T)$, where $W_e(T)$ is the essential numerical range of T ([12], [16]). Then

$$(*) \qquad r_e(T)\leqq w_e(T)\leqq \|\pi(T)\|.$$

We recall that $W_e(T)$ can be defined by the following identities ([12, Lemma 3. 3):

$$(**) \qquad W_e(T) = \bigcap_{K\in\mathcal{K}} \overline{W(T+K)} = \bigcap_{(1-Q)\in\mathcal{P}_f} \overline{W(T_Q)},$$

where $W(S)$ represents the numerical range of S.

iv) It is easy to give examples of operators S for which $r_e(S)<w_e(S)<\|\pi(S)\|$. We can take, for instance, S to be the 2×2 scalar operator matrix

$$S=\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}$$

acting on $\mathfrak{H}\oplus\mathfrak{H}$ in the usual fashion. We see that $r_e(S)=0$, $w_e(S)=1$, $\|\pi(S)\|=2$. It is also easy enough to find an operator T for which $r_e(T)=w_e(T)<\|\pi(T)\|$. Let us observe first that for every $R_1, R_2\in\mathscr{L}(\mathfrak{H})$ $W_e(R_1\oplus R_2)$ coincides with the convex hull of $W_e(R_1)\cup W_e(R_2)$ (this readily follows from $(**)$). Now, let R be any operator such that $r_e(R)=\|\pi(R)\|=1$ (take for instance $R=1$) and set $T=R\oplus S$, where S is the nilpotent operator defined previously. Then

$$\|\pi(T)\|=\max\left(\|\pi(R)\|, \|\pi(S)\|\right)=2,$$

while, by the preceding comment, $w_e(T)=1$ and hence $1=r_e(T)=w_e(T)<\|\pi(T)\|=2$.

We are indebted to J. STAMPFLI who has pointed out to us that the remaining situation concerning the strict inequality in $(*)$ is impossible. The proof of this fact, that we present below, is a simplification of Stampfli's argument.

Lemma 2. 3. *Let $T\in\mathscr{L}(\mathfrak{H})$ and suppose that $\|\pi(T)\|=w_e(T)$. Then $w_e(T)=r_e(T)$.*

Proof. Let $\lambda\in W_e(T)$ be such that $|\lambda|=\|\pi(T)\|$. It can be easily proved ([12], Lemma 2. 1) that

$$(***)\qquad \|\pi(T)\|=\inf_{(1-Q)\in\mathscr{P}_f}\|TQ\|.$$

Define, inductively, an orthonormal sequence $\{x_n\}$ in $\mathfrak{H}$ and a decreasing sequence of projections $\{Q_n\}$ in $\mathscr{L}(\mathfrak{H})$ as follows: let x_0 be any unit vector in $\mathfrak{H}$ and Q_0 be any projection in $\mathscr{L}(\mathfrak{H})$ such that $Q_0x_0=0$ and $(1-Q_0)\in\mathscr{P}_f$; having defined x_k and Q_k for $0\leqq k\leqq n$, let $x_{n+1}\in Q_n\mathfrak{H}$ with $\|x_{n+1}\|=1$ and let $Q_{n+1}\leqq Q_n$ with $Q_{n+1}x_{n+1}=0$, $(1-Q_{n+1})\in\mathscr{P}_f$ such that $|(Tx_{n+1}, x_{n+1})-\lambda|\leqq 1/n$, $\|TQ_{n+1}\|\leqq|\lambda|+1/(n+1)$ (the existence of x_{n+1} and Q_{n+1} is guaranteed by conditions $(**)$ and $(***)$). Since $|(Tx_n, x_n)|\leqq\|Tx_n\|\leqq|\lambda|+1/(n-1), n>1$ and $|(Tx_n, x_n)|\to|\lambda|$ it follows that $\|Tx_n\|\to|\lambda|$. Also, we see that $\|(T-\lambda)x_n\|^2=\|Tx_n\|^2-\bar\lambda(Tx_n, x_n)-\lambda\overline{(Tx_n, x_n)}+|\lambda|^2\to 0$. If $\pi(T-\lambda)$ were invertible, then there would exist $S\in\mathscr{L}(\mathfrak{H})$ such that $\pi(S)\pi(T-\lambda)=\pi(1)$ and hence $S(T-\lambda)=1+K$ for some $K\in\mathscr{K}$; but $(S(T-\lambda)x_n, x_n)\to 0$ while $((1+K)x_n, x_n)\to 1$. Therefore $\lambda\in E(T)$ and hence $w_e(T)=|\lambda|\leqq r_e(T)$.

3. Upper-semicontinuity of the essential spectrum. Let $\mathscr{B}$ be a complex Banach algebra with identity. It is well known ([10]) that the spectrum $\Sigma(T)$ of $T\in\mathscr{B}$ is an upper-semicontinuous function of T. The next lemma shows that each separated part (closed and open subset) is also an upper-semicontinuous function of T.

Theorem 3.[2]) *For $T\in\mathscr{B}$, let Σ be a non-empty clopen (closed and open) subset $\Sigma(T)$, and set $\Sigma' = \Sigma(T)-\Sigma$. If V and V' are two disjoint neighborhoods of Σ and Σ' respectively, then there exists $\varepsilon>0$ such that for every $S\in\mathscr{B}$ with $\|T-S\| < \varepsilon$ the following conditions are satisfied:*

a) $\Sigma(S) \subset V\cup V'$, and the set $\Lambda = \Sigma(S)\cap V$ is not empty,

b) if E_Σ and E_Λ are the idempotents associated with Σ and Λ corresponding to T and S, then there exists a constant $k>0$ such that $\|E_\Sigma - E_\Lambda\| < k\|T-S\| < 1$, $1\leqq j\leqq n$,

*c) if **B** is the Banach algebra of all bounded operators on a complex Banach space, then* ran E_Σ *is topologically isomorphic to* ran E_Λ.

Proof. Let W be an open subset of $\mathbf{C}$ such that $\Sigma\subset W$, $\overline{W}\subset V$ and $\Gamma = \overline{W}-W$ consisting of finitely many rectifiable closed Jordan curves. From the upper-semicontinuity of $\Sigma(T)$ there exists $\delta>0$ such that if $\|T-S\| < \delta$, then $\Sigma(S) \subset W\cup V'$. Thus $\Sigma(S)\cap V = \Sigma(S)\cap W = \Lambda$. Let $M = \sup_{\lambda\in\Gamma}\|(T-\lambda)^{-1}\|$. Since $M<\infty$ we can choose $0<\eta<\min(\delta, 1/M)$ so that, if $\|T-S\| < \eta$, then $\sup_{\lambda\in\Gamma}\|(T-\lambda)-(S-\lambda)\| < \inf_{\lambda\in\Gamma}(1/\|(T-\lambda)^{-1}\|)$ and hence

$$\sup_{\lambda\in\Gamma}\|(S-\lambda)^{-1}\| \leqq \sup\frac{\|(T-\lambda)^{-1}\|}{1-\|T-S\|\,\|(T-\lambda)^{-1}\|} \leqq \frac{M}{1-\eta M}$$

for every $S\in\mathscr{B}$ with $\|T-S\| < \eta$. Employing the last inequality we obtain

$$\|E_\Sigma - E_\Lambda\| = \left\|(1/2\pi i)\int_\Gamma [(T-\lambda)^{-1}-(S-\lambda)^{-1}]\,d\lambda\right\| \leqq$$

$$\leqq (1/2\pi)\int_\Gamma \|(T-\lambda)^{-1}(T-S)(S-\lambda)^{-1}\|\,|d\lambda| \leqq k\|T-S\|,$$

with $k = \dfrac{M^2}{2\pi(1-\eta M)}\displaystyle\int_\Gamma |d\lambda|$. Choosing $0<\varepsilon<\min\left(\eta, \dfrac{1}{k}\right)$ part b) follows. Since the norm of any non-zero idempotent is never less than one, a) is also clear. Finally, part c) is a consequence of the following classical result ([11]).

Lemma (Sz.-Nagy). *If F and G are idempotents on a Banach space and $\|F-G\| < 1$, then* ran F *and* ran G *are topologically isomorphic, and hence they have the same dimension.*

Theorem 4. *Each separated part (clopen subset) of a) $E(T)$, b) $B(T)$, c) $\Omega(T)$ is an upper-semicontinuous function of $T\in\mathscr{L}(\mathfrak{H})$, in the sense of Theorem 3.*

[2]) A slightly different version of this result in case $\mathfrak{B}$ is equal to the algebra of all bounded operators on a Banach space can be found in [8], Theorem 3. 16 (Chap. 4).

Proof. Part a) follows directly from Lemma 3. 1 taking $\mathscr{B}=\mathscr{L}(\mathfrak{H})/\mathscr{K}$, and the fact that $\pi:\mathscr{L}(\mathfrak{H})\to\mathscr{L}(\mathfrak{H})/\mathscr{K}$ is norm decreasing.

In order to prove b) assume that B is a non-void clopen subset of $B(T)$ and write $B'=B(T)-B$. Suppose also that V and V' are two disjoint open neighborhoods of B and B', respectively. We must show that there exists $\eta>0$ such that for every $S\in\mathscr{L}(\mathfrak{H})$ with $\|T-S\|<\eta$ we have $B(S)\subset V\cup V'$ and $B(S)\cap V\neq\emptyset$. In fact taking smaller neighborhoods if necessary we can assume $\Sigma(T)\cap b(V\cup V')=\emptyset$. Since $\Sigma(T)-[\overline{V\cup V'}]$ consists of finitely many points we can choose a neighborhood V'' of $\Sigma(T)-[\overline{V\cup V'}]$ such that $V''\cap(V\cup V')=\emptyset$. From Theorem 1 $E(T)\cap V\neq\emptyset$ and $E(T)\subset V\cup V'$. Now using Theorem 3 and part a) we conclude that there exists $\eta>0$ such that, if $S\in\mathscr{L}(\mathfrak{H})$ and $\|T-S\|<\eta$, then $E(S)\subset V\cup V'$, $\Sigma(S)\subset V\cup V'\cup V''$ and $E(S)\cap V\neq\emptyset$. It follows, again from Theorem 1, that $B(S)\cap V''=\emptyset$ and hence that $B(S)\subset V\cup V'$, $B(S)\cap V\neq\emptyset$.

Finally to prove c) suppose that Ω is a non-void clopen subset of $\Omega(T)$ and let $\Omega'=\Omega(T)-\Omega$. Also, let U and U' be two disjoint open neighborhoods of Ω and Ω' respectively. Furthermore, let W and W' be relatively compact open neighborhoods of Ω and Ω' such that $\overline{W}\subset U$, $\overline{W}'\subset U'$. Since the set Φ_0 of all Fredholm operators of index zero is open, there exists $\delta'>0$ such that for any $S\in\mathscr{L}(\mathfrak{H})$ with $\|T-S\|<\delta'$ we have $S-\lambda\in\Phi_0$, for every $\lambda\in b(W\cup W')$. On the other hand, from part a) there exists $\varepsilon>0$ such that $\|T-S\|<\varepsilon$ implies $E(S)\subset W\cup W'$ and $E(S)\cap W\neq\emptyset$. It follows from Theorem 1 that for every $S\in\mathscr{L}(\mathfrak{H})$ with $\|T-S\|<\delta=\min(\varepsilon,\delta')$ we have $\Omega(S)\subset W\cup W'$ and $\Omega(S)\cap W=\Omega(S)\cap U\neq\emptyset$. This completes the proof of the theorem.

Given $T\in\mathscr{L}(\mathfrak{H})$ we say that an invariant (closed) subspace of T is hyperinvariant ([5]) if it is invariant under every operator in the commutant $\mathscr{A}'_T$ of T (recall that $\mathscr{A}'_T=\{S\in\mathscr{L}(\mathfrak{H}):ST=TS\}$). Let $\mathscr{A}''_T$ be the double commutant of T, i.e. $\mathscr{A}''_T=\{R\in\mathscr{L}(\mathfrak{H}):RS=SR$, for all $S\in\mathscr{A}'_T\}$. Clearly $T\in\mathscr{A}''_T\subset\mathscr{A}'_T$.

It is easy to check that for every $R\in\mathscr{A}''_T$ the range of R and the null of R are hyperinvariant subspaces of T.

Theorem 5. *If $T\in\mathscr{L}(\mathfrak{H})$ and $E(T)$ is disconnected, then there exists $\varepsilon>0$ such that for every $R\in\mathscr{L}(\mathfrak{H})$, $\|R\|<\varepsilon$ and for every $K\in\mathscr{K}$, the operator $T+R+K$ has a non-trivial hyperinvariant subspace.*

Proof. The theorem is a direct consequence of Theorem 4-a) and the next lemma.

Lemma 3. 1. *For $T\in\mathscr{L}(\mathfrak{H})$, let Σ be a clopen subset $\Sigma(T)$ and let E_Σ be the associated idempotent, then* ran E_Σ *and* null E_Σ *are hyperinvariant subspaces of T. Furthermore, if $E(T)$ is disconnected then T has a non-trivial hyperinvariant subspace.*

Proof. The first part follows from the fact that E_Σ belongs to the rational algebra generated by T and hence to $\mathscr{A}''_T$. To prove the second part note that if $\Sigma(T)$ is connected, then there exists $\lambda_0 \in \Sigma(T)-E(T)$ such that either ran $(T-\lambda_0)$ is proper or null $(T-\lambda_0)$ is so. In any case T has a non-trivial hyperinvariant subspace, as asserted.

In section 4 we will see that if $T\in(ED)$, then a more precise version of Theorem 5 can be given.

Two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of $\mathfrak{H}$ are said to be complementary if there exists an idempotent $F\in\mathscr{L}(\mathfrak{H})$ such that $\mathfrak{M}=\operatorname{ran} F$, $\mathfrak{N}=\operatorname{null} F$.

Theorem 6. *Let m, n be two non-zero cardinal numbers such that $m+n = \dim\mathfrak{H}$, then the set of all operators in $\mathscr{L}(\mathfrak{H})$ having two complementary hyperinvariant subspaces of dimension m and n has a non-void interior.*

Proof. Let $P\in\mathscr{L}(\mathfrak{H})$ be an (orthogonal) projection such that $\dim \operatorname{ran} P=m$, $\dim \operatorname{null} P=n$. It is easy to see that $P=(-1/2\pi i)\int_{|\lambda-1|=1/2}(P-\lambda)^{-1}\,d\lambda$ (where the circle $|\lambda-1|=1/2$ is positively oriented). From Theorem 3 we can find an $\varepsilon>0$ such that, if $\|S-P\|<\varepsilon$, then $\Lambda=\Sigma(S)\cap\{\lambda\in\mathbf{C}:|\lambda-1|<1/2\}\neq\emptyset$ is a proper clopen subset of $\Sigma(S)$ and $\dim \operatorname{ran} E_\Lambda=m$, $\dim \operatorname{null} E_\Lambda=n$, where

$$E_\Lambda=(-1/2\pi i)\int_{|\lambda-1|=1/2}(S-\lambda)^{-1}\,d\lambda.$$

The theorem follows, now, from Lemma 3. 1.

Theorem 7. *The class (ED) is uniformly open in $\mathscr{L}(\mathfrak{H})$.*

Proof. The following elementary topological lemma together with Theorem 4-a) show that each separated part of $\hat{E}(T)$ is an upper-semicontinuous function of T. Clearly, from this assertion, the theorem follows.

Lemma 3. 2. *Let X be a compact subset of the plane and let U be an open neighborhood of $\hat{X}$, then there exists an open neighborhood V of X such that if Y is any compact subset of V, it follows that $\hat{Y}\subset U$.*

4. On invariant subspaces of operators in *(ED)*. In this paragraph we turn our attention to the proof of a decomposition theorem for the invariant subspace lattice $\mathscr{I}_T$ of an operator $T\in(ED)$. The techniques provided in [4] and [5] are basic for our purposes. We start our discussion with a lemma which is useful for proving that certain operators lie in (*ED*).

Lemma 4. 1. *Let $T\in\mathscr{L}(\mathfrak{H})$ and assume that Σ is a clopen subset of $\Sigma(T)$. If E_Σ denotes the associated idempotent, then $E(T)\cap\Sigma\neq\emptyset$ if and only if* ran E_Σ *is* *infinite dimensional. In this case* $E(T)\cap\Sigma = E[T|\text{ran } E_\Sigma]$. *Thus, if* $E(T)\cap\Sigma = \emptyset$, Σ *consists only of finitely many points which are eigenvalues of finite multiplicity.*

Proof. Assume ran E_Σ is finite dimensional and let $\lambda_0\in\Sigma$, write $S = T-\lambda_0$, $S_1=S|\text{ran } E_\Sigma$ and $S_2=S|\text{null } E_\Sigma$. Since $\Sigma(S_2) = \Sigma(T)-\Sigma$, S_2 is invertible. Therefore ran S = ran S_1+ran S_2. But ran S_2=null E_Σ is closed and ran S_1 if finite dimensional, then ran S is also closed. Since null $S\subset$ran E_Σ and ran $S\supset$ $\supset$null E_Σ we conclude that $S(=T-\lambda_0)$ is a Fredholm operator, and hence $\lambda_0\notin E(T)$. Conversely, suppose that ran E_Σ is infinite dimensional. If null E_Σ is finite dimensional, from the first part of the proof, $E(T)\subset\Sigma$ and there is nothing to prove. Thus, we can also assume that null E_Σ is infinite dimensional. At this point we need the following auxiliary construction: let $\mathfrak{H}_1$=ran E_Σ, $\mathfrak{H}_2=\mathfrak{H}_1^\perp$, also let $Z_1\in\mathscr{L}(\mathfrak{H}_1)$ be the identity operator on $\mathfrak{H}_1$, and let $Z_2:\mathfrak{H}_2\to$null E_Σ be the bounded linear transformation given by $Z_2 = (1-E_\Sigma)|\mathfrak{H}_2$. It is easy to see that Z_2 is invertible (it is bijective). Define now the invertible transformation Z: $\mathfrak{H}_1\oplus\mathfrak{H}_2 \to \mathfrak{H}$ by $Z|\mathfrak{H}_1=Z_1$, $Z|\mathfrak{H}_2=Z_2$. Letting $T_1=T|\text{ran } E_\Sigma$, $T_2=T|\text{null } E_\Sigma$ and observing that $Z^{-1}TZ = Z_1^{-1}T_1Z_1\oplus\oplus Z_2^{-1}T_2Z_2$ we have $E(T) = E(Z^{-1}TZ) = E(Z_1^{-1}T_1Z_1)\cup E(Z_2^{-1}T_2Z_2) = E(T_1)\cup E(T_2)$. Since $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are infinite dimensional we conclude that $E(T_j)= E(Z_j^{-1}T_jZ_j)\neq\emptyset$, $j=1, 2$, and hence $E(T)\cap\Sigma = E(T_1) \neq \emptyset$, $E(T)\cap\cap[\Sigma(T)-\Sigma] = E(T_2) \neq \emptyset$. The proof of the lemma is complete.

Given $T\in\mathscr{L}(\mathfrak{H})$, let $\mathscr{I}_T$ be the lattice of all invariant subspaces of T with the topology induced by the distance between subspaces, namely, if P, Q are the (orthogonal) projections onto the subspaces $\mathfrak{M}, \mathfrak{N}\in\mathscr{I}_T$, then $\Theta(\mathfrak{M},\mathfrak{N}) = \|P-Q\|$. It can be proved that ([5], Corollary 1. 2) if $\mathfrak{M}$ is an isolated point of $\mathscr{I}_T$, then it is a hyperinvariant subspace of T.

Theorem 8. *Let* $T\in(ED)$, *that is assume there exist two proper clopen subsets* Λ_1 *and* Λ_2 *of* $E(T)$ *such that* $E(T) = \Lambda_1\cup\Lambda_2$ *and* $\hat\Lambda_1\cap\hat\Lambda_2 = \emptyset$. *Then there exist two infinite dimensional complementary subspaces* $\mathfrak{M}_1, \mathfrak{M}_2\in\mathscr{I}_T$ *which are isolated points of* $\mathscr{I}_T$, *such that if* $T_j=T|\mathfrak{M}_j$, $j=1, 2$ *then* $\mathscr{I}_T$ *is homeomorphic to the topological product* $\mathscr{I}_{T_1}\times\mathscr{I}_{T_2}$ *and* $E(T_j)=\Lambda_j$, $j=1, 2$.

Proof. From Theorem 1, $\hat\Sigma(T)-\hat E(T)$ is a set of isolated points, thus there exist two proper clopen subsets Σ_1, Σ_2 of $\Sigma(T)$ such that $\Sigma_1\cup\Sigma_2 = \Sigma(T)$, $\hat\Sigma_1\cap\cap\hat\Sigma_2 = \emptyset$ and $\Lambda_j\subset\Sigma_j$, $j=1, 2$. Now, let $\mathfrak{H}_1$=ran E_{Σ_1}, $\mathfrak{H}_2=\mathfrak{H}_1^\perp$. Furthermore, let $\mathfrak{M}_j$=ran E_{Σ_j}, and $T_j=T|\mathfrak{M}_j$, $j=1, 2$. We deduce, as in the proof of Lemma 4. 1 that there exist an invertible transformation $Z:\mathfrak{H}_1\oplus\mathfrak{H}_2 \to \mathfrak{H}$ such that $Z^{-1}TZ = Z_1^{-1}T_1Z_1\emptyset Z_2^{-1}T_2Z_2$. Using now Theorems 2 and 6 of [5] we obtain $\mathscr{I}_T \approx\approx\mathscr{I}_{Z^{-1}TZ}\approx\mathscr{I}_{Z_1^{-1}T_1Z_1}\times\mathscr{I}_{Z_2^{-1}T_2Z_2}\approx\mathscr{I}_{T_1}\times\mathscr{I}_{T_2}$. On the other hand, using Theorem 1 in [4], it is easy to check that the subspaces $\mathfrak{H}_1\oplus\{0\}$ and $\{0\}\oplus\mathfrak{H}_2$ are isolated points of $\mathscr{I}_{Z^{-1}TZ}$ and hence $\mathfrak{M}_1, \mathfrak{M}_2$ are isolated points of $\mathscr{I}_T$. The last part of the theorem follows from Lemma 4. 1.

We close this section with a couple of results that illustrate how to produce nontrivial examples of operators in (ED).

Theorem 9. *Let P be any (orthogonal) projection in $\mathscr{L}(\mathfrak{H})$ and let V be any isometry. Then the 2×2 self-adjoint operator matrix*

$$T = \begin{bmatrix} P & V \\ V^* & 0 \end{bmatrix},$$

acting in the usual fashion on $\mathfrak{H}\oplus\mathfrak{H}$, is in (ED). Furthermore, if $\Lambda_1=[-1, 1/2(1-\sqrt{5})]$, $\Lambda_2 = [0, 1/2(1+\sqrt{5})]$, then $\Sigma(T) \subset \Lambda_1 \cup \Lambda_2$ and $E(T)\cap \Lambda_j \neq \emptyset$ $(j=1, 2)$.

Proof. Let p be the cubic polynomial $p(\lambda) = \lambda^3-\lambda^2-\lambda+1$. It can be checked that

$$p(T) = \begin{bmatrix} 1-(P-VV^*)^2 & 0 \\ 0 & V^*PV \end{bmatrix}.$$

Since it is clear that V^*PV and $(P-VV^*)^2$ are positive contractions, it follows that $p(T)$ enjoys the same property. Therefore from the spectral mapping theorem $\Sigma(T) \subset p^{-1}[0, 1] = \Lambda_1 \cup \Lambda_2$. Now, let $\mathfrak{M}_1(\mathfrak{M}_2)$ be the range of the map $y \to \begin{bmatrix} Vy \\ y \end{bmatrix} \left(y \to \begin{bmatrix} -Vy \\ y \end{bmatrix}\right)$ from $\mathfrak{H}$ into $\mathfrak{H}\oplus\mathfrak{H}$. It can be easily checked that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are infinite dimensional orthogonal subspaces and that the compression of T to $\mathfrak{M}_1(\mathfrak{M}_2)$ is a positive invertible (negative invertible) operator. We use this preceding remark to prove that $E(T)\cap \Lambda_j \neq \emptyset$. This will clearly complete the proof of the theorem. Assume for example that $E(T)\cap \Lambda_1 = \emptyset$. Then (Lemma 4. 1) $\Sigma(T)\cap \Lambda_1$ consists of finitely many points which are eigenvalues of finite multiplicity. Since T is self-adjoint there exists a finite rank projection $Q\in \mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$ such that $TQ(=QT)$ is positive and $T(1-Q)$ is negative. Let T' and T'' be the compressions of TQ and $T(1-Q)$, respectively to $\mathfrak{M}_1$. Then T' and $-T''$ are positive operators and $T' = T_1+(-T'')$ (where T_1 is the compression of T to $\mathfrak{M}_1$) is a positive invertible operator on the infinite dimensional space $\mathfrak{M}_1$. This is a contradiction since T' is a compact operator. An analogous reasoning shows that $E(T)\cap$ $\cap \Lambda_2 \neq \emptyset$.

Theorem 10. *Let $A\in \mathscr{L}(\mathfrak{H})$ be such that $\hat{E}(A)$ does not touch the segment $\Lambda=[-2, 2]$ of the real axis. Then the operator $T\in \mathscr{L}(\mathfrak{H}\oplus\mathfrak{H})$ defined by*

$$T = \begin{bmatrix} A & 1 \\ -1 & 0 \end{bmatrix}$$

is in (ED).

Proof. Using [7], Problem 55, it readily follows that $\Sigma(T)=f^{-1}[\Sigma(A)]$, where f is the analytic function defined on $\mathbf{C}-\{0\}$ by $f(\lambda) = \lambda+1/\lambda$. Furthermore, we see that $E(T)=f^{-1}[E(A)]$. Also it can be checked that $f^{-1\wedge}[E(A)]=f^{-1}[\hat{E}(A)]$. Therefore $f^{-1}[\hat{E}(A)]=\hat{E}(T)$. Since f maps the exterior of the closed unit disc D onto $\mathbf{C}-A$ and the interior of $D-\{0\}$ onto the same region, the theorem follows.

5. Open questions. As the reader may have noticed this paper raises some interesting questions. For example, from the proof of Lemma 2. 2, the following inclusion formula can be obtained:

$$\Omega(T)\subset \bigcap_{(1-Q)\in\mathcal{P}_f} \Sigma(T_Q)\subset B(T),$$

where $\mathcal{P}_f$ is the set of all finite rank projections and $T_Q=(QT)|\text{ran } Q$. Therefore it is natural to ask: does there exist $T\in\mathcal{L}(\mathfrak{H})$ for which the inclusions in the above chain of inequalities become proper?

Moreover, we can also ask the following related question: if $(1-Q)\in\mathcal{P}_f$ and $T\in\mathcal{L}(\mathfrak{H})$, does $\Sigma(T)-\Sigma(T_Q)$ consist only of isolated points?

Observe that if the last statement holds, then $B(T)=\bigcap_{(1-Q)\in\mathcal{P}_f}\Sigma(T_Q)$, as it is easy to verify.

In a different direction we may ask: do there exist compact operators K_j, $0\leqq j\leqq 4$, such that the following conditions are satisfied?

a) $\Sigma(T+K_0)=\Omega(T)$, b) $B(T+K_1)=\Omega(T)$, c) $\Sigma(T+K_2)=B(T)$,
d) $\Sigma(T+K_3)=B(T+K_3)$, e) $\hat{\Sigma}(T+K_4)=\hat{B}(T+K_4)(=\hat{B}(T))$.

Furthermore,

f) if $T\in\mathcal{L}(\mathfrak{H})$ and $\hat{\Sigma}(T+K)$ is disconnected, for every $K\in\mathcal{K}$, is T in (ED)?

Note that a⇒b and (a *or* c)⇒d⇒e⇒f. On the other hand (b *and* c)⇒a.

Finally we give some fragmentary results in the positive direction concerning these last questions.

Theorem 11. *Let $T\in\mathcal{L}(\mathfrak{H})$ and assume that all points in $\Sigma(T)-B(T)$, except for a finite number of them, are reducing eigenvalues of T (i.e. the corresponding eigenspaces are reducing subspaces of T), then there exists $K\in\mathcal{K}$ such that $\Sigma(T+K)=B(T)$.*

Proof. If $\Sigma(T)-B(T)=\emptyset$, there is nothing to prove. Let λ_n, $n=1,2,\dots$ be the points in $\Sigma(T)-B(T)$ and let ν_n, $n=1,2,\dots$ be complex numbers such that the points $\mu_n=\nu_n+\lambda_n$ lie in the set $\Sigma'(T)$ of limit points of $\Sigma(T)$, and $\inf_{\mu\in\Sigma'(T)}|\lambda_n-\mu|=|\nu_n|$. Also, let $A_n=\Sigma(T)-\bigcup_{k=1}^{n}\{\lambda_k\}$ and $E_{\{\lambda_n\}}$, E_{A_n} be the idempotents associated with $\{\lambda_n\}$, A_n; define $K_m=\sum_{n=1}^{m}\nu_n E_{\{\lambda_n\}}$. Then

$$T+K_m=(T+K_m)E_{A_m}+(T+K_m)\left(\sum_{n=1}^{m}E_{\{\lambda_n\}}\right)=TE_{A_m}+\sum_{n=1}^{m}\{\mu_n+N_n\}E_{\{\lambda_n\}},$$

where N_n is a nilpotent operator acting on ran $E_{\{\lambda_n\}}$, $1 \leq n \leq m$. Therefore $\Sigma(T+K_m) = \Lambda_m \cup \bigcup_{n=1}^{m} \{\mu_n\} = \Lambda_m$. This completes the proof of the theorem in case $\Sigma(T) - B(T)$ is finite. By hypothesis there exists m_0, such that if $m > m_0$, then $E_{\{\lambda_m\}}$ is an orthogonal projection and since $E_{\{\lambda_m\}} E_{\{\lambda_n\}} = 0$, for $n \neq m$ and $\lim_{n\to\infty} \nu_n = 0$, we see that K_m converges, in the norm topology, to a compact operator K such that $TK = KT$. Since $T+K$ commutes with $T+K_m$, for each $m = 1, 2, \ldots$ we have ([8], Chap. IV, Theorem 3. 6) $\Sigma(T+K) = \lim_{m\to\infty} \Sigma(T+K_m) = \lim_{m\to\infty} \Lambda_m = B(T)$. Here the limits are taken in the Hausdorff metric topology for compact subsets of the plane.

Remark. i) We point out that the actual hypothesis needed to prove the preceding theorem is that the idempotent $E_{\{\lambda\}}$ be selfadjoint for all, but a finite number of $\lambda \in \Sigma(T) - B(T)$. On the other hand, BROWDER proved in [2], § 6, Lemma 17, that $\lambda \in \Sigma(T) - B(T)$ if and only if λ is an isolated eigenvalue of finite multiplicity of T which is a pole of the resolvent function $\mu \to (\mu - T)^{-1}$, $\mu \notin \Sigma(T)$. Also, it is shown in [6], Chap. 7, § 3, Theorem 18, that the residuum of the resolvent function around a pole $\lambda \in \Sigma(T)$ is $E_{\{\lambda\}}$. Therefore, the requirement stated at the begining of the present remark is equivalent to the following growth condition: if $\lambda \in \Sigma(T) - B(T)$ and m is the order of the pole λ,

$$\lim_{\mu\to\lambda} \left\| \frac{1}{(m-1)!} \frac{d^{(m-1)}}{d\mu^{(m-1)}} (\mu-\lambda)^m (\mu - T)^{-1} \right\| = 1 (= \|E_{\{\lambda\}}\|).$$

ii) All points in $B(T) - \Omega(T)$ are eigenvalues of finite multiplicity, but none of them are reducing. In fact, an elementary argument shows that λ is a reducing eigenvalue of T if and only if null $(T^* - \bar{\lambda}) \subset$ null $(T - \lambda)$. Suppose that λ is reducing and $(T-\lambda) \in \Phi_0$; since dim null $(T-\lambda) =$ dim null $(T^* - \bar{\lambda})$, then null $(T-\lambda) =$ null $(T^* - \bar{\lambda})$. This implies that $(T-\lambda)|[\text{null}\,(T-\lambda)]^\perp$ is invertible and hence λ is an isolated point of $\Sigma(T)$. We conclude that $\lambda \notin B(T)$, as asserted.

iii) Let X be a subset of the plane and let conh X denote its convex hull. From Theorem 1-c it readily follows that conh $E(T) =$ conh $\Omega(T) =$ conh $B(T)$.

As a consequence of the next theorem we shall obtain more information about the convex hull of the essential spectrum of T in case T is hyponormal (i.e. $\|T^* x\| \leq \|Tx\|$, for all $x \in \mathfrak{H}$).

Theorem 12. *If $T \in \mathscr{L}(\mathfrak{H})$ is hyponormal, then there exists a normal compact operator K such that $T+K$ is hyponormal and $\Sigma(T+K) = \Omega(T)$.*

Proof. It is well known that if T is hyponormal, then every eigenvalue of T is reducing (null $(T^* - \bar{\lambda}) \subset$ null $(T-\lambda)$, for all $\lambda \in \mathbf{C}$). Therefore, from the preceding remark ii), $\Omega(T) = B(T)$. Let K be the compact operator constructed in the proof of Theorem 11. Arguing as in [3], Corollary 3. 3 we see that K is normal and $T+K$ is hyponormal. The second assertion follows directly from Theorem 11.

Corollary 5. 1. *If T is hyponormal, then* conh $E(T)=W_e(T)$. *Moreover, there exists $K\in\mathcal{K}$ such that* $\overline{W(T+K)}=W_e(T)$.

Proof. Let $K\in\mathcal{K}$ as in Theorem 12. Then from [15], Theorem 2 we have $\overline{W(T+K)}=\text{conh}\,\Sigma(T+K)=\text{conh}\,\Omega(T)=\text{conh}\,E(T)\subset W_e(T)$. Since $W_e(T)$ is clearly contained in $\overline{W(T+K)}$, the proof is complete.

Corollary 5. 2. *If T is hyponormal, then* $r_e(T)=\|\pi(T)\|$.

Proof. Let K as in Theorem 12, then from [14], Theorem 1, $\|\pi(T)\|\geqq r_e(T)=$ $=\text{rad}\,\Omega(T)=\text{rad}\,\Sigma(T+K)=\|T+K\|\geqq\|\pi(T)\|$, and the assertion follows.

Added in proof. The first question in Sec. 5 has a negative answer while the second question has an affirmative one. The solution to these problems are included in the author's recent paper "A characterization of the Browder spectrum", to appear in the *Proc. Amer. Math. Soc.* Problems a) and b) (and hence problems c), d), e) and f)) have been answered in the affirmative by J. G. STAMPFLI.

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(Received March 24, 1971)